ADDING NUMBERS

- DETERMINENTAL POINT PROCESSES
- ADDITIVE COMBINATORICS
- WE CAN FIND MATH ANYPLACE

\[
\begin{array}{c}
7.7 \\
8.5 \\
8.3 \\
5.8 \\
1.9 \\
6.5 \\
2.7 \\
\hline
37
\end{array}
\]
EASY STUFF

1. **Point Process on** \( \{1, 2, \ldots, n-1\} \)
   \( P \text{ on } 2^{\mathcal{A}} \), "pick \( A \subseteq \{1, 2, \ldots, n-1\} \)

2. Set \( x_i = \mathbb{1} \) as i \( \in \mathcal{A} \) or not

   For base \( \mathcal{B} \) carries
   \[ P\{x_i = 1\} = \frac{1}{2} - \frac{1}{2b} \in \left( \frac{1}{b}, \frac{3}{4b} \right) = \{ v \} \text{ for } b = 10 \]
   \[ P\{x_i = x_2 = \ldots = x_{n-1} = 1\} = \frac{1}{b} \text{ so } \text{cov}(x_i, x_2) = -\frac{1}{b} \left( 1 - \frac{1}{b} \right) \]

3. \( \{x_i\} \) is **stationary**

4. \( \{x_i\} \) is **one-dependent**: \( x_i \cdot x_{i+1} \perp x_{i+2} \ldots x_{n-1} \)

5. **So all standard limit theorems**:

   \( \mathcal{H} \) carries \( T_n \sim \text{normal}, \left( \sigma^{-1} x^\frac{1}{2} - \frac{1}{2} \right), \frac{\gamma_n}{\sigma^2} \left( 1 - \frac{1}{b} \right) \)
DETERMINANTS

\[ P(x_1 = x_1, \ldots, x_n = x_n) = \frac{1}{b^n} \text{DET} \left( \frac{S_i + S_j + b - 1}{b - 1} \right) \]

\[ x_i = 1 \text{ AT } S = \{ s_x, s_y \} \text{, (add) \times (add), } S_0 = 0, S_\infty = n \]

EX \[ n = 8, \ S = S_\infty = 1, \ 1000 \ 000 \]

\[ \frac{1}{b^n} \left( \begin{array}{ccc}
(3 + h) & (4 + h) & (5 + h) \\
4 + h & (5 + h) & (6 + h) \\
0 & 1 & (2 + h)
\end{array} \right) \]

\[ S_\infty, b = 2 \cdot 0.85, \ b = 10 \cdot 0.0104 \]

WHY? \[ \lambda_n(S) = \# \ \text{SEQ DESC } \leq S \]
\[ R_n(S) = \# \ \text{SEQ DESC } \leq S \]

\[ \lambda_n(S) = \sum_{T \subseteq S} P_n(T), \quad P_n(S) = \sum_{T \subseteq S} \text{WT} \lambda_n(T) \]

\# \ \text{WK \ SEQ LENGTH L IS } \binom{S + b - 1}{b - 1} \]
\[ \lambda_n(S) = \binom{S + b - 1}{b - 1} \binom{S - 2 + b - 1}{b - 1} \]

\[ \sum_{T \subseteq S} \text{WT} = \binom{S + b - 1}{b - 1} \]

\[ E \ S = \emptyset \ P(\text{NO CARRIES}) = \frac{\binom{S + b - 1}{b - 1}}{b^n}, \ n = 8, \ b = 16 = 0.002 \]
DETERMINANTAL POINT PROCESSES

Let \( P(A) = \mathbb{P}\{X_i = 1, i \leq A\} \) "A-point correlations."

\[
\text{DEF } \{X_i\}_i \text{ is determinantal if } \exists K(x,y):
\]

\[
P(A) = \det(K_{i,j})_{i,j=1}^A
\]

EXAMPLES

. Eigenvalues of random matrices
  e.g. CUE: Pick \( M \in \mathbb{U}_n \), \( K(0,0') = \frac{\sin \theta_{0,0'}}{\pi(\theta_{0,0'})^2} \)
  Neutron scattering, antenna design, Riemann H

. Edges in MST in a graph

. Random analytic functions
  \[
  X_i = \sum_{j=0}^{13} Z_j 3^i \quad 13l = 1, \quad Z_j \sim \mathcal{N}(0,1)
  \]
  \[
  K(3,4) = \frac{1}{\pi(1-\rho^2)}
  \]
Quantum Mechanics

\[ \psi_i(x, \tau) \] "state" (= prob elec. at \( x \) at time \( \tau \))

In noninteracting fermions states \( \psi_1, \ldots, \psi_n \),

anti symmetrize \( \prod \psi_i(x, \tau) \) to

\[ \sum_{\pi \in S_n} \text{sgn}(\pi) \prod \psi_{\pi_i}(x, \tau) \]

\[ = \text{det} K(x, \bar{x}), \quad K(x, y) = \sum_{i=1}^n \psi_i(x) \bar{\psi}_i(y) \]

Carries are determinental

\[ K(x, x) = \delta(x-x), \quad \sum_{x \in \mathcal{X}} \Delta x^2 = \frac{L}{1-(1-3)^2} \]

Many more examples, in general spaces

\[ P\{X=x\} = K(x, x), \quad \text{cov}(X_i, X_j) = \text{det} \left( K_{ij} \right) - k(x, y) \]

If \( K_n(x, y) \to K(x, y) \) then \( X \Rightarrow X \)

Easier to describe \( K(x, y) \) then Pm 2

Research Problem: Given \( x \), estimate \( K \)
BACK TO CARRIES

LET $C_b$ be cyclic group

$C_b \leq C_b^2$ as $0, 5, 10, 15, \ldots (b-1)b$

CHOOSE COSET REPRESENTATIVES

\[
\begin{pmatrix}
0 & \ldots & 9 \\
0 & \ldots & 9 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 9
\end{pmatrix}
\]

(9) CARRIES

ARE THERE DIFFERENT DIGITS, FEWER CARRIES

E.G. $b = 5$, $C_5 \leq C_{25}$ \{0, 5, 10, 15, 20\},

USE $0 \pm 1, \pm 1$ BALANCED REPS

\[
\begin{pmatrix}
-2 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

6 CARRIES

VERSUS 10 FOR

REPS 0, 1, 2, 3, 4, 5
EX \( b=5 \), 'digits 0,±1,±2, write -1=1, etc

\[
\begin{array}{c}
2 \\
2 \\
2 \\
1 \\
1 \\
1 \\
\hline
2
\end{array}
\]

Again if 'digits' on left are

\[
\begin{array}{c}
2 \\
2 \\
2 \\
1 \\
1 \\
5 \\
\hline
2
\end{array}
\]

'i.e.d. digits on right are too

'carry' of \( 5 \) if \( \rightarrow 1 \)

\( 5 \) if \( \rightarrow 1 \)

\[
\begin{array}{c}
2 \\
2 \\
2 \\
1 \\
1 \\
5 \\
\hline
2
\end{array}
\]

Any choice of 'digits' leads to 1-dependent determinental process

(Indeed, coset reps for any \( H \triangleleft G \), \( H \neq \{e\} \))

Computer designers know about 'balance'

(look up balanced ternary in {Knuth VolD})

Usual coset reps \( \{0,1,\ldots, b-1\} \)

\( \text{carry} = \frac{1}{2} \)

Balanced \( \Rightarrow \{0,±1,±(\frac{b}{2})\} \)

\( \Rightarrow \text{V4} \)

Can we do better?

(Why must we have at least some carries)
ANSWERS

A) NOPE, for $2 \leq b \leq 10$ (COMPUTER)

B) NOPE, if $b = p$ a suff. large prime

😢, ALAS, PF USES ADDITIVE COMBINATORICS

So $p \gg 2^{2^n}$ TOWER OF LENGTH $\ell^5$

TH (WITH SOUND SHAB): $\forall \epsilon > 0 \exists p \in \text{PRIMES} \rightarrow$ ANY CHOICE OF COSET REPS $c_p \leq c_p$

HAS AT LEAST $p^2(\frac{A}{2} - \epsilon)$ CARRIES.

IDEA

1) PF EASY FOR $A \leq 2$, SAY $A = \{0, \epsilon, q, \epsilon q\}$

· IF $q_i + q_j = q_k$ AT MOST $A$ CHOICES FOR $i$

· So $\#\text{SOLS} \leq \sum_{i=1}^{2^A} I = \frac{2^A(2^A+1)}{2} \leq \frac{1}{2} 1A^2 + 1A$

· GENERAL $A \#\text{SOLS} = \frac{3}{2} 1A^2 + 1A$

· BEST is $A = 0, 1, 2, 3, \ldots, \frac{2\ell}{3}$
1) For $A \leq C^*$ use $\text{BALOG SZEMEREDI TO SHOW}$

$$\forall \epsilon > 0 \exists L \exists$$

1. $A = A_0 U A_1 \ldots U A_L$
2. $A_i$ 'LARGE' ($|A_i| \geq \epsilon |A|$)
3. $A_i$ 'STRUCTURED' ($1A_i + A_i \leq \epsilon |A_i|$)
4. $A_i, A_j$ 'RANDOM w.r.t. EACH OTHER'

$$E(A_i, A_j) \leq \epsilon |A_i|^2 |A_j|^2$$

5. $A_0$ 'RANDOM' ($E(A_0, A) \leq \epsilon |A|^2$)

$$(E(A_i, A) = H_{a, a', b, b'}: a + a' = b + b', \text{eq.})$$

$$(E(A, A) = 1A_1^3 \notin) A \text{ IS A SUBGR}$$

3) Then $A_i$ is a subset of $\mathbb{Z}$, use easy ans.

$A_0$ doesn't matter.

"Can't somebody find an easy pf?"
REFERENCES

1. THE BULK OF THIS TALK CAN BE FOUND IN BORODIN, A., DIACONIS, P. AND FULMAN, J. (2009) "ON ADDING A LIST OF NUMBERS (AND OTHER 1-DEPENDENT DETERMINATIONAL POINT PROCESSES)" BULL. AMER. MATH. SOC. 47, 639-670. THIS CONTAINS POINTERS TO DETERMINATIONAL POINT PROCESSES.
