ADDING NUMBERS

- DETERMINENTAL POINT PROCESSES
- ADDITIVE COMBINATORICS
- WE CAN FIND MATH ANYPLACE

$$\begin{array}{rcc}
7 & \cdot & 7 \\
8 & \cdot & 5 \\
8 & & 3 \\
5 & & 8 \\
1 & \cdot & 9 \\
6 & & 5 \\
2 & & 7 \\
\hline
37 & &
\end{array}$$

EASY STUFF

- POINT PROCESS ON $\{1, 2, \cdots, n-1\}$
  P ON $2^{[n-1]}$, " PICK $A \subseteq \{1, 2, \cdots, n-1\}$

- SET $X_i = \begin{matrix} 1 \\ 0 \end{matrix}$ AS $i \in A$ OR NOT

  FOR BASE b CARRIES

  $$P\{X_i = 1\} = \frac{1}{2} - \frac{1}{2b} \left(= \frac{\binom{b}{2}}{b^2}\right) = \begin{cases} .45 & b = 10 \\ .25 & b = 2 \end{cases}$$

- $P(X_1 = X_2 = \cdots = X_{i-1} = 1) = \frac{\binom{b}{i}}{b^i}$ SO $\mathrm{cov}(X_1, X_2) = -\frac{1}{12}\left(1 - \frac{1}{b^2}\right)$

- $\{X_i\}$ IS STATIONARY

- $\{X_i\}$ IS ONE-DEPENDENT: $X_1 \cdots X_{i-1} \perp X_{i+1} \cdots X_{n-1}$

  SO ALL STANDARD LIMIT THEOREMS:

  \# CARRIES $T_{n-1} \sim$ NORMAL, $(n-1)\left(\frac{1}{2} - \frac{1}{2b}\right), \frac{n+1}{12}\left(1 - \frac{1}{b^2}\right)$

DETERMINANTS

$$P\{X_1=e_1,\dots,X_{n-1}=e_{n-1}\} = \frac{1}{b^n}\,\mathrm{DET}\begin{pmatrix} s_{j+1}-s_i+b-1 \\ b-1\end{pmatrix}$$

$e_i = 1$ AT $S=\{s_1<s_2\cdots<s_k\}$, $(k+1)\times(k+1)$, $s_0=0$, $s_{k+1}=n$

EX $n=8$ $s_1=s_5=1$ 1000100

$$\frac{1}{b^8}\begin{pmatrix} \binom{4+b-1}{b-1} & \binom{5+b-1}{b-1} & \binom{8+b-1}{b-1} \\ 1 & \binom{4+b-1}{b-1} & \binom{7+b-1}{b-1} \\ 0 & 1 & \binom{3+b-1}{b-1} \end{pmatrix}$$

EG, $b=2$ .0352, $b=10$ .0104

WHY? $\alpha_n(S)$ = # SEQ DESC $\subseteq S$

$\beta_n(S)$ = " $= S$

$$\alpha_n(S) = \sum_{T\subseteq S}\beta_n(T), \quad \beta_n(S) = \sum_{T\subseteq S}(-1)^{|S-T|}\alpha_n(T)$$

WK ↑ SEQ LENGTH ℓ IS $\binom{\ell+b-1}{b-1}$ $\underbrace{-\,-\,-\,-\,-\,-\,-\,-}_{\ell+b-1}$

$$\alpha_n(S) = \binom{s_1+b-1}{b-1}\binom{s_2-s_1+b-1}{b-1}\cdots\binom{n-s_k+b-1}{b-1}$$

EG $S=\emptyset$ P(NO CARRIES) $= \dfrac{\binom{n+b-1}{b-1}}{b^n}$, $n=8$, $b=10$ $= .0002$

DETERMINENTAL POINT PROCESSES

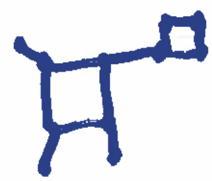
LET $\rho(A) = P\{X_i = 1, i \in A\}$ "k-POINT CORRELATIONS"

<u>DEF</u> $\{X_i\}_{i=1}^{n}$ IS DETERMENTAL IF $\exists\, K(x,y)$:

$$\rho(A) = \det_{x,y \in A} \left(K(x,y) \right)$$

EXAMPLES

- EIGENVALUES OF RANDOM MATRICES
  eg CUE: PICK $M \in \mathcal{U}_n$,  $K(\theta,\theta') = \frac{\sin n(\theta-\theta')}{\pi(\theta-\theta')}$
  NEUTRON SCATTERING, ANTENA DESIGN, RIEMANN H

- EDGES IN MST IN A GRAPH 

- RANDOM ANALYTIC FUNCTIONS

$$f(z) = \sum_{j=0}^{\infty} Z_j z^j \quad |z|<1, \quad Z_j \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

 $K(z,w) = \frac{1}{\pi(1-z\bar{w})^2}$

• QUANTUM MECHANICS

$\psi_i(x,t)$ "STATE" (= PROB ELEC. AT x AT TIME t)

n NONINTERACTING FERMIONS STATES $\psi_1 \cdots \psi_n$,

ANTI SYMMETRIZE $\prod \psi_i(x_i,t)$ TO

$$\sum_{\sigma \in S_n} \operatorname{SGN}(\sigma) \prod_{i}^{n} \psi_{\sigma(i)}(x_i,t)$$

$$= \operatorname{DET} K(x_i, x_j), \quad K(x,y) = \sum_{i=1}^{n} \psi_i(x) \bar{\psi}_i(y)$$

• CARRIES ARE DETERMINENTAL

$$K(i,j) = a(j-i), \quad \sum_{n \in \mathbb{Z}} a(n) z^n = \frac{1}{1-(1-z)^b}$$

MANY MORE EXAMPLES, IN GENERAL SPACES

• $P\{X_x = 1\} = K(x,x)$, $\operatorname{COV}(X_x, X_y) = \operatorname{DET}\begin{pmatrix} \cdots \end{pmatrix} - K(x,x)K(y,y)$

• IF $K_n(x,y) \to K(x,y)$ THEN $X^n \Rightarrow X$

• EASIER TO DESCRIBE $K(x,y)$ THEN P ON $2^{\mathcal{X}}$.

<u>RESEARCH PBLM</u> GIVEN X, ESTIMATE K

BACK TO CARRIES

LET C_b BE CYCLIC GROUP

$C_b \leq C_{b^2}$ AS $0, b, 2b, 3b, \dots (b-1)b$

CHOOSE COSET REPRESENTATIVES

eg $C_{10} \leq C_{100}$ REP $0,1,2,\dots,9$

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & \cdots & & 9 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ 9 \end{matrix} & \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & & 1 \\ 0 & 0 & \cdots & 1 & 1 \\ & & & & \\ 0 & 1 & 1 & 1 \cdots & 1 \end{pmatrix} \end{array}$$

$\binom{b}{2}$ CARRIES

ARE THERE DIFFERENT DIGITS, FEWER CARRIES

EG. $b=5$, $C_5 \leq C_{25}$ $\{0, 5, 10, 15, 20\}$.

USE $0 \pm 1, \pm 2$ BALANCED REPS

$$\begin{array}{c|c} & \begin{matrix} -2 & -1 & 0 & 1 & 2 \end{matrix} \\ \begin{matrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 20 & 20 & 0 & 0 & 0 \\ 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 & 5 \end{pmatrix} \end{array}$$

6 CARRIES

VERSUS 10 FOR

REPS $\{0,1,2,3,4\}$

<u>EX</u> $b=5$, 'DIGITS $0, \pm 1, \pm 2$, WRITE $-1 = \underline{1}$, ETC

$\underline{2}$	$\underline{5}$	$\underline{2}$
$\underline{2}$		1
$\underline{2}$		$\underline{1}$
1		0
1	5	1
2		$\underline{2}$
$\underline{2}$		

<u>AGAIN</u> IF 'DIGITS' ON LEFT ARE i.i.D., DIGITS ON RIGHT ARE TOO

'CARRY' OF $\underline{5}$ IF $\begin{matrix} -2 \to 1 \\ -1 \to 2 \end{matrix}$

5 IF $\begin{matrix} 1 \to -2 \\ 2 \to -1 \end{matrix}$

- <u>ANY</u> CHOICE OF 'DIGITS' LEADS TO 1-DEPENDENT DETERMINENTAL PROCESS
- (INDEED, COSET REPS FOR ANY $H \trianglelefteq G$ ($H \leq Z(G)$
- COMPUTER DESIGNERS KNOW ABOUT 'BALANCE (LOOK UP BALANCED TERNARY IN $\left\{ \begin{matrix} \text{KNUTH VOL II} \\ \text{WIKIPEDIA} \end{matrix} \right.$)

- USUAL COSET REPS $\{0, 1, \cdots, b-1\}$ CARRY $\doteq \frac{1}{2}$

BALANCED " $\{0, \pm 1, \cdots \pm (\frac{b-1}{2})\}$ " $\doteq 1/4$

CAN WE DO BETTER?

(WHY MUST WE HAVE AT LEAST SOME CARRIES

ANSWERS

A) NOPE, FOR $2 \le b \le 10$ (COMPUTER)

B) NOPE, IF $b = p$ A SUFF. LARGE PRIME

☹, ALAS, PF USES ADDITIVE COMBINATORICS

SO $p \gg 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ TOWER OF LENGTH $1/\epsilon^5$

TH (WITH SOUND, SHAO): $\forall\, \epsilon > 0\ \exists\, p^* \ni p > p^*$

$\Rightarrow$ ANY CHOICE OF COSET REPS $C_p \subseteq C_{p^2}$

HAS AT LEAST $p^2(\frac{1}{4} - \epsilon)$ CARRIES.

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IDEA

1) PF EASY FOR $A \subseteq \mathbb{Z}$, SAY $A = \{0 < a_1 < a_2 \cdots < a_n\}$

- IF $a_i + a_j = a_k$ AT MOST k CHOICES FOR i
- SO $\#$ SOLS $\le \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \le \frac{1}{2}|A|^2 + |A|$
- GENERAL A $\#$ SOLS $\le \frac{3}{4}|A|^2 + |A|$
- BEST IS $A = 0, \pm 1, \pm 2, \cdots \pm \frac{n-1}{2}$

2) FOR $A \subseteq C_{p^2}$ USE $\left\{\begin{array}{l}\text{BALOG} \\ \text{SZEMEREDI} \\ \text{GOWERS}\end{array}\right.$ TO SHOW

$\forall \epsilon > 0 \; \exists \, L \ni$

- $A = A_0 \cup A_1 \cup \cdots \cup A_L$
- A_i 'LARGE' $(\# A_i \geq \epsilon |A|)$ $\quad 1 \leq i \leq L$
- A_i 'STRUCTURED' $(|A_i + A_i| \leq \tilde{\epsilon} |A_i|)$
- A_i, A_j 'RANDOM WRT EACH OTHER'

$$E(A_i, A_j) \leq \epsilon |A_i|^{\frac{3}{2}} |A_j|^{\frac{3}{2}}$$

- A_0 'RANDOM' $(E(A_0, A) \leq \epsilon |A|^3)$

($E(A,B) = \#\ a, a', b, b' : a + a' = b + b'$, eg
eg $E(A,A) = |A|^3 \Leftrightarrow A$ is A SUBGP.)

3) THEN $A_i \overset{F}{\simeq}$ SUBSET OF $\mathbb{Z}$, USE EASY ARG.

A_0 DOESN'T MATTER.

?CAN'T SOMEBODY FIND AN EASY PF?

REFERENCES

1. THE BULK OF THIS TALK CAN BE FOUND IN BORODIN, A., DIACONIS, P. AND FULMAN, J. (2009) "ON ADDING A LIST OF NUMBERS (AND OTHER 1-DEPENDENT DETERMINENTAL POINT PROCESSES) BULL. AMER. MATH SOC. 47, 639–670. THIS CONTAINS POINTERS TO DETERMINENTAL POINT PROCESSES.

2. A BIBLE ON ADDITIVE COMBINATORICS IS: TAO, T. AND VU, V. (2006) ADDITIVE COMBINATORICS. A GENTLE INTRODUCTION IS IN B. GREEN'S REVIEW OF THE BOOK

3. THE WORK ON ADDITIVE COMBINATORICS AND 'BALANCED DIGITS' IS FROM: DIACONIS, P. SOUNDARAJAN, K. AND SHAO, X. (2012) 'CARRIES AND ADDITIVE COMBINATORICS FORTHCOMING.