Xampling at Sub-Nyquist Rates: Correlations, Nonlinearities, and Bounds

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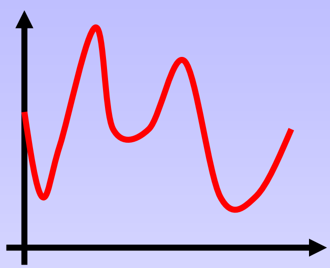
In collaboration with my students at the Technion

Sampling: “Analog Girl in a Digital World…” Judy Gorman 99

Analog world

Digital world



$x(t)$

Sampling ADC

$c[n]$

- Signal processing
- Image denoising
- Analysis…

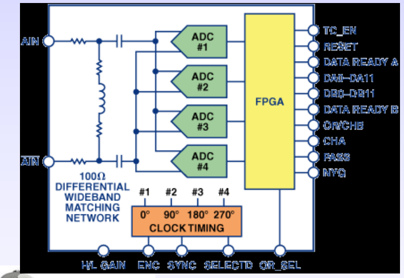
$\tilde{x}(t)$

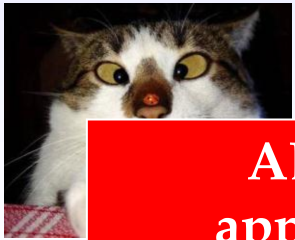
Reconstruction DAC

- Music
- Radar
- Image…

- Very … tes: hard … solutions

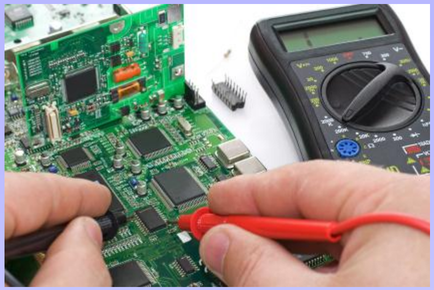
- High DSP rates





ADCs, the front end of every digital application, remain a major bottleneck

Today's Paradigm



The Separation Theorem:

- Circuit designer experts design samplers at Nyquist rate or higher
- DSP/machine learning experts process the data
  - Typical first step: Throw away (or combine in a "smart" way e.g. dimensionality reduction) much of the data …
  - Logic: Exploit structure prevalent in most applications to reduce DSP processing rates
  - DSP algorithms have a long history of leveraging structure: MUSIC, model order selection, parametric estimation …
  - However, the analog step is one of the costly steps

Can we use the structure to reduce sampling rate + first DSP rate (data transfer, bus …) as well?

Key Idea

Exploit analog structure to improve processing performance:

- Reduce storage/reduce sampling rates
- Reduce processing rates
- Reduce power consumption
- Increase resolution
- Improve denoising/deblurring capabilities
- Improved classification/source separation

Goal:

- Survey the main principles involved in exploiting analog structure
- Provide a variety of different applications and benefits

Talk Outline

- Motivation
- Classes of structured analog signals
- Xampling: Compression + sampling
- Sub-Nyquist solutions
  - Multiband communication: Cognitive radio
  - Time delay estimation: Ultrasound, radar, multipath medium identification
- Ultrasound and compressed beamforming
- Nonlinear compressed sensing: Phase retrieval

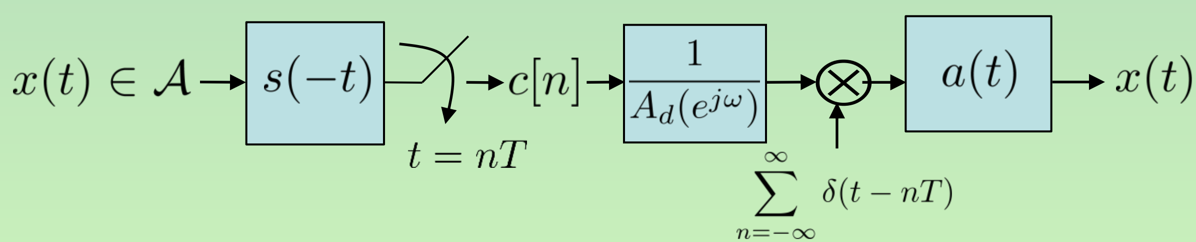
Classical/Modern Sampling Theory

- Sampling theory has developed tremendously in the 60+ years since Shannon
- Many beautiful results, and many contributors

(Unser, Aldroubi, Vaidyanathan, Blu, Jerri, Vetterli, Grochenig, Feichtinger, DeVore, Daubechies, Christensen, Eldar, …)

- Recovery methods have been developed for signals in arbitrary subspaces
- Recovery from nonlinear samples as well (*Landau, Mirenker and Sandberg 60's for bandlimited inputs, Dvorkind, Matusiak and Eldar 2008 arbitrary subspaces and filters)*

Perfect Reconstruction in a Subspace

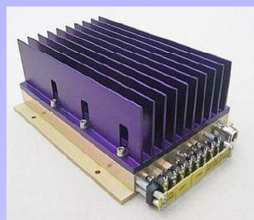


- Subspace prior $\quad x(t) = \sum_{n=-\infty}^{\infty} d[n] a(t-nT)$
- Recovery filter

$$A_d(e^{j\omega}) = \mathbb{F}\left\{\langle a(t), s(t-nT)\rangle\right\} = \sum_{k=-\infty}^{\infty} A(\omega/T + 2\pi k) S^*(\omega/T + 2\pi k) \qquad (\mathcal{A} \oplus \mathcal{S}^{\perp} = \mathcal{L}_2)$$

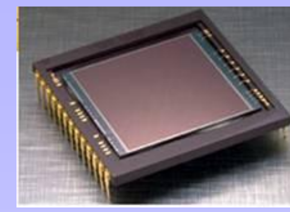
Nonlinear Sampling

Power amplifiers

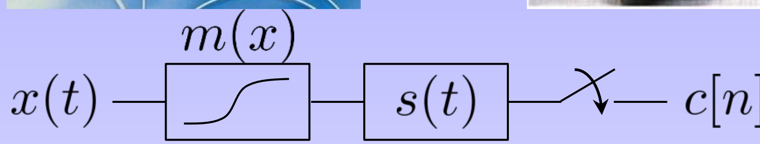


Optical modulators

CCD arrays



Companding



Theorem (Uniqueness)

Assume that $\mathcal{A} \oplus \mathcal{S}^{\perp} = \mathcal{L}_2$. If m is invertible and its derivative m' satisfies

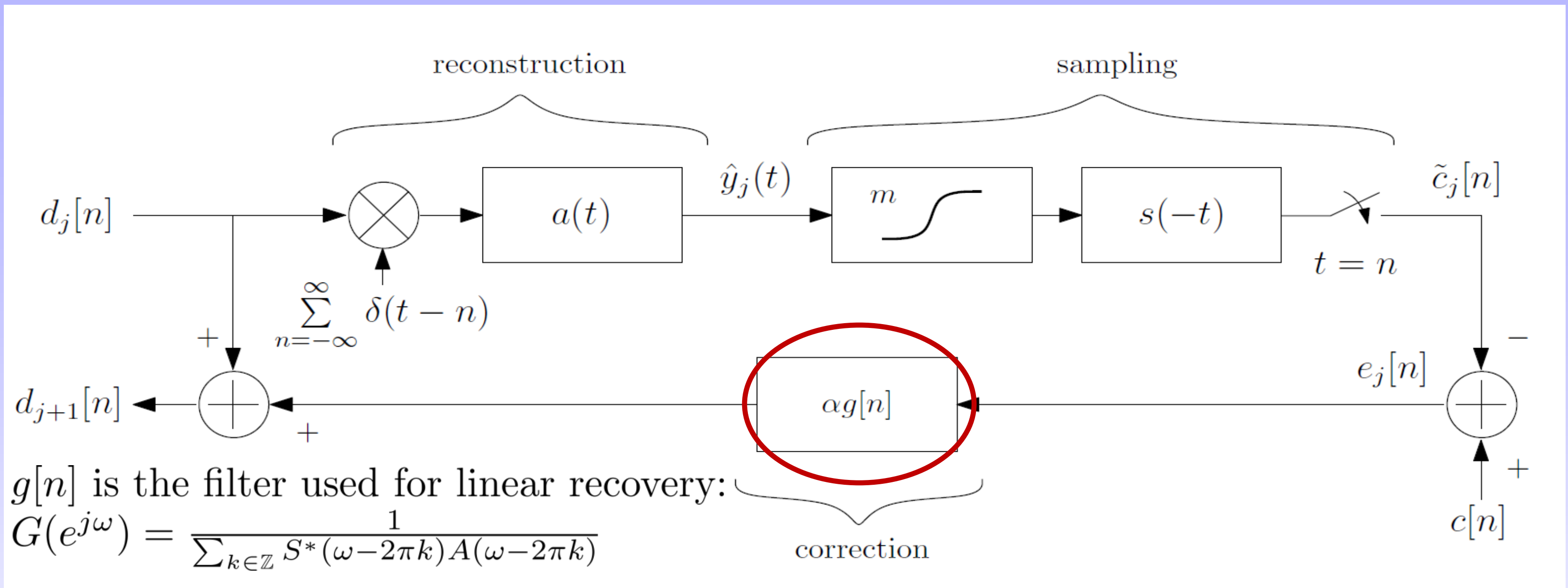
$$\frac{\inf_x m'(x)}{\sup_x m'(x)} > \sin(\mathcal{A}, \mathcal{S})$$

then there is a unique $\hat{x}(t) \in \mathcal{A}$ consistent with the samples $c[n]$. Furthermore, the objective $\|S^* m(x) - c\|_2$ has a single stationary point.

(Dvorkind, Eldar & Matusiak, 08)

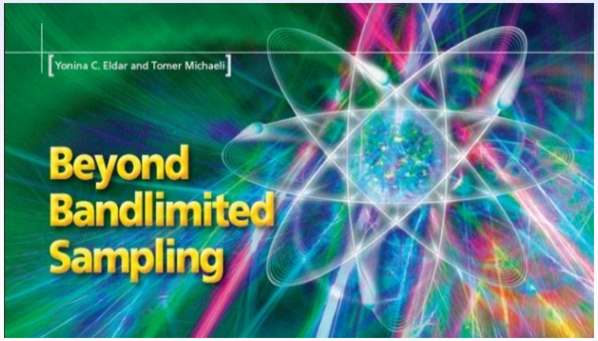
- Since there is a unique stationary point any appropriate descent method will converge to the true input

Recovery from Nonlinear Samples



$g[n]$ is the filter used for linear recovery: $G(e^{j\omega}) = \frac{1}{\sum_{k\in\mathbb{Z}} S^*(\omega-2\pi k)A(\omega-2\pi k)}$

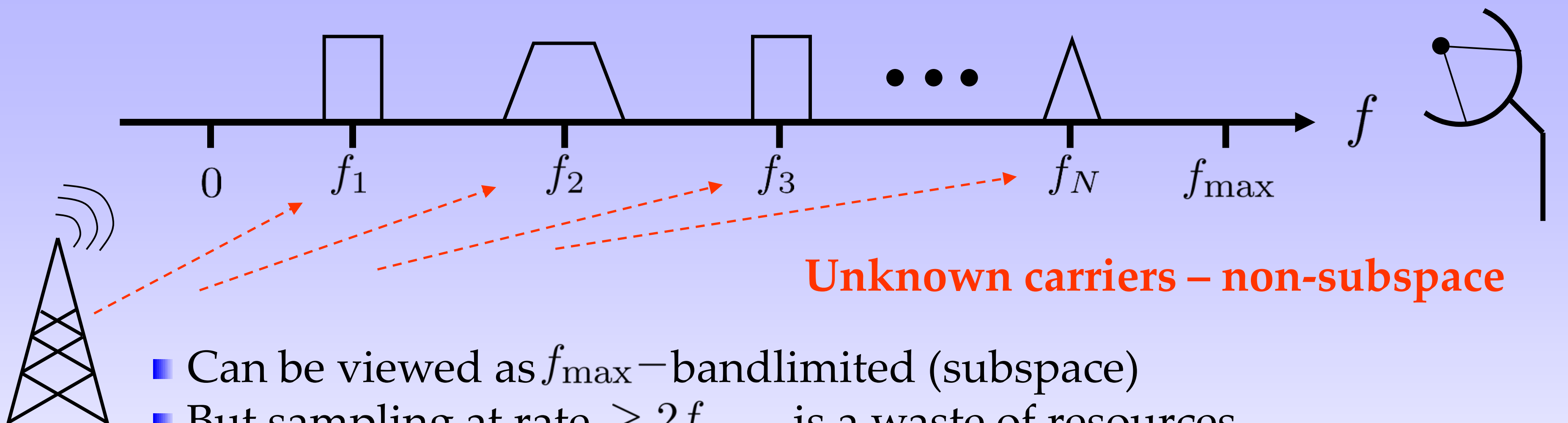
More information:
Y. C. Eldar and T. Michaeli, "Beyond Bandlimited Sampling," *IEEE Signal Proc. Magazine*, 26(3): 48-68, May 2009



Structured Analog Models

Multiband communication:

(Landau,Scott,White,Vaughan,Kohlenberg,Lin,Vaidyanathan,Herley,Wong,Feng,Bresler,Mishali,Eldar …)



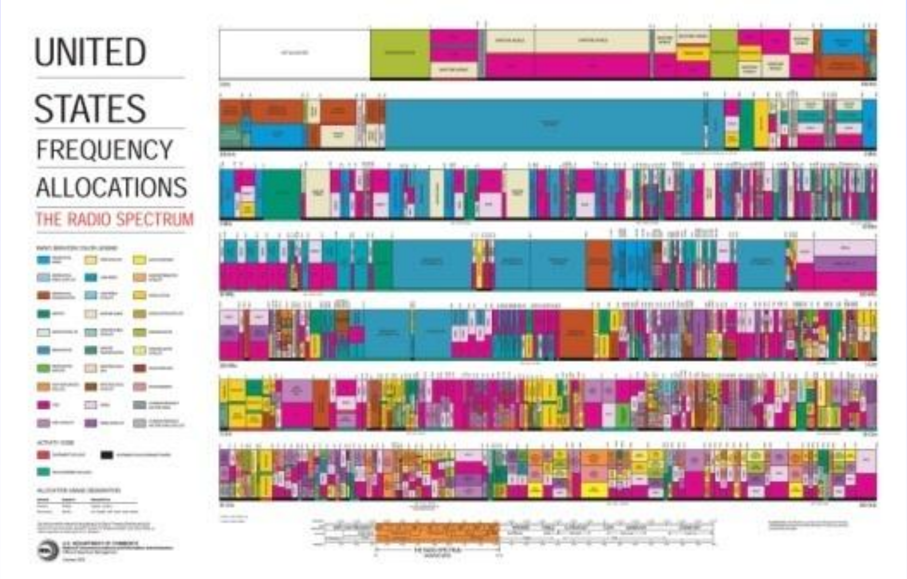
- Can be viewed as $f_{\max}-$bandlimited (subspace)
- But sampling at rate $\geq 2f_{\max}$ is a waste of resources
- For wideband applications Nyquist sampling may be infeasible
- Previous work either assumes known carriers or uses samplers with Nyquist-rate analog bandwidth

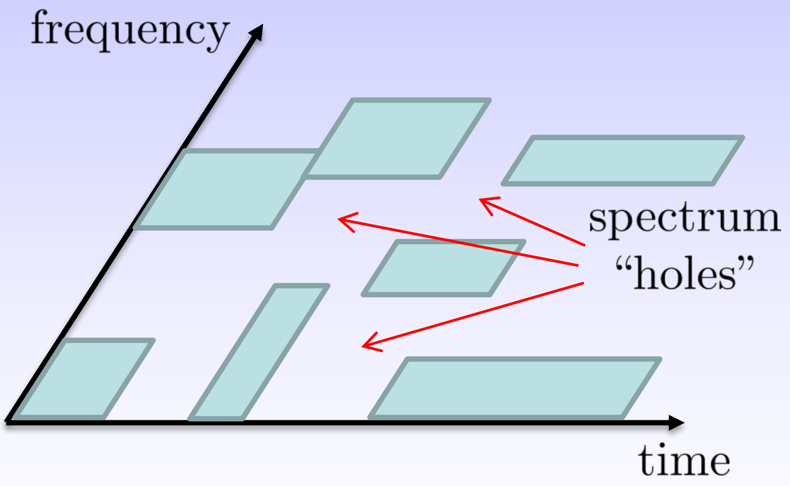
Question:
How do we treat structured (non-subspace) models efficiently?

Cognitive Radio

- Cognitive radio mobiles utilize unused spectrum ``holes''
- Spectral map is unknown a-priori, leading to a multiband model

Federal Communications Commission (FCC) frequency allocation

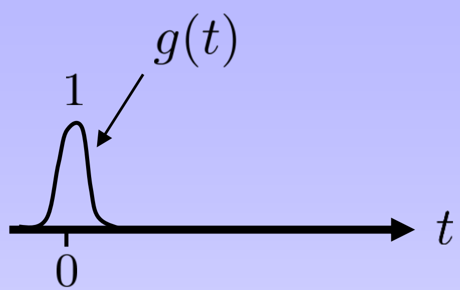


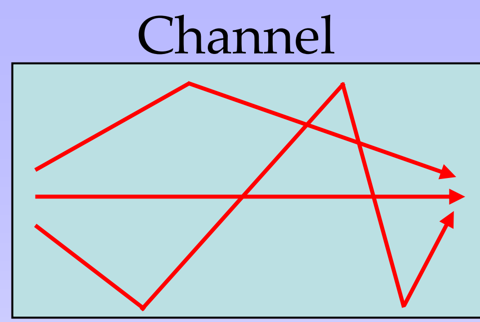


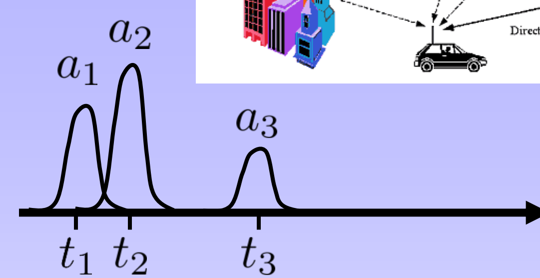
Licensed spectrum highly underused: E.g. TV white space, guard bands and more

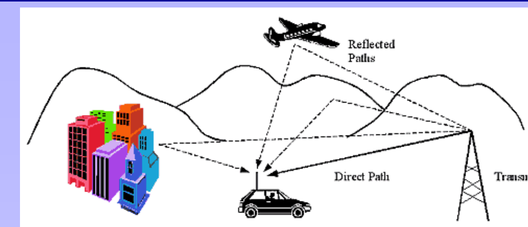
Structured Analog Models

Medium identification:









Similar problem arises in radar, UWB communications, timing recovery problems …

Unknown delays – non-subspace

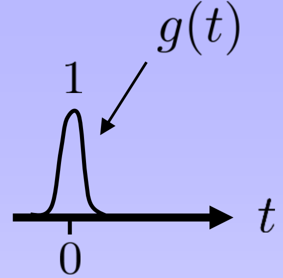
- Digital match filter or super-resolution ideas (MUSIC etc.) *(Quazi,Brukstein, Shan,Kailath,Pallas,Jouradin,Schmidt,Saarnisaari,Roy,Kumaresan,Tufts …)*
- But requires sampling at the Nyquist rate of $g(t)$
- The pulse shape is known – No need to waste sampling resources!

Question (same):
How do we treat structured (non-subspace) models efficiently?

Ultrasound

- High digital processing rates
- Large power consumption

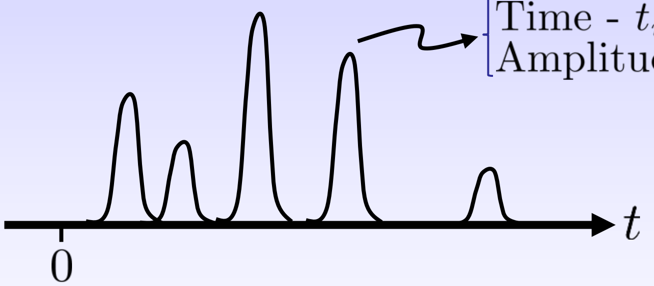
Tx pulse



Ultrasonic probe

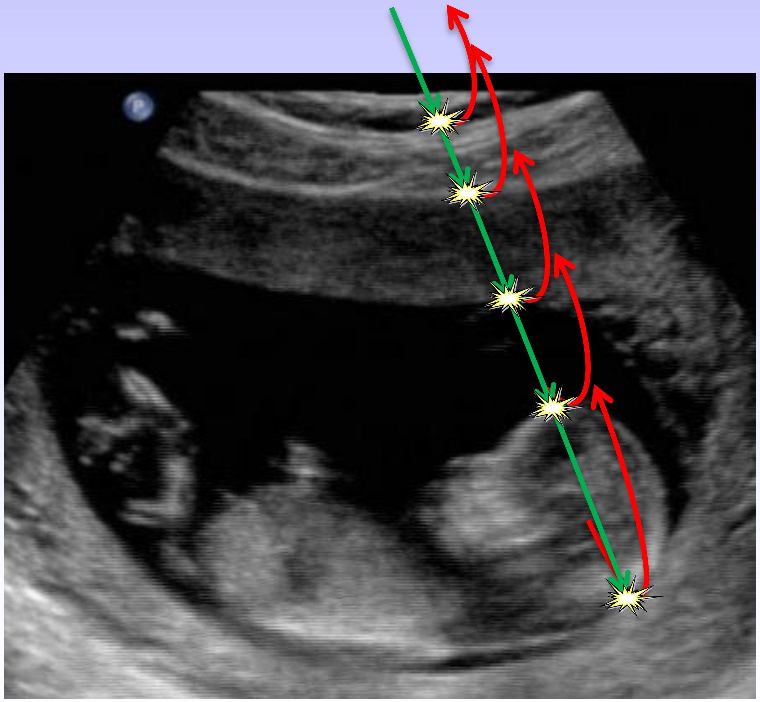
(Collaboration with General Electric Israel)

Rx signal



Unknowns

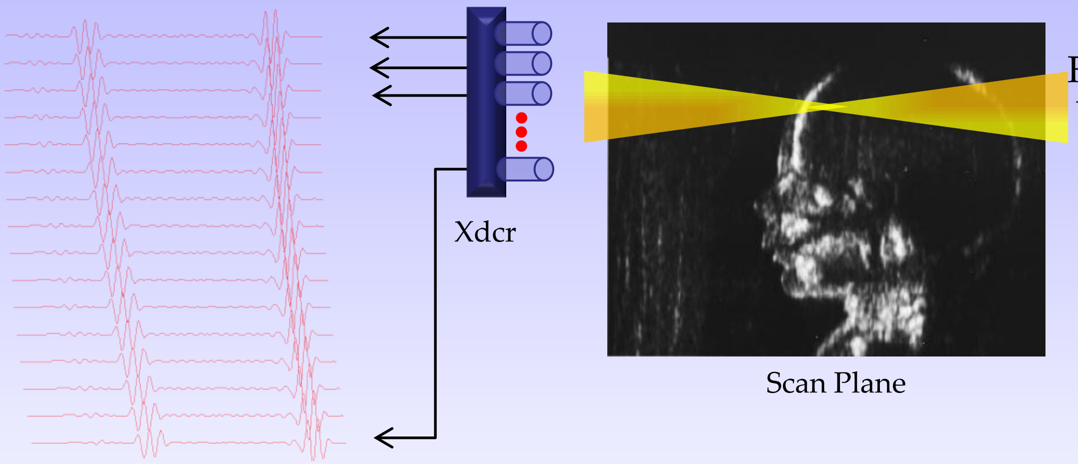
Time - t_i
Amplitude - a_i



- Echoes result from scattering in the tissue
- The image is formed by identifying the scatterers

Processing Rates

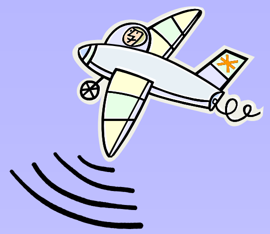
- To increase SNR the reflections are viewed by an antenna array
- SNR is improved through beamforming by introducing appropriate time shifts to the received signals



- Requires high sampling rates and large data processing rates
- One image trace requires 128 samplers @ 20M, beamforming to 150 points, a total of 6.3×10^6 sums/frame

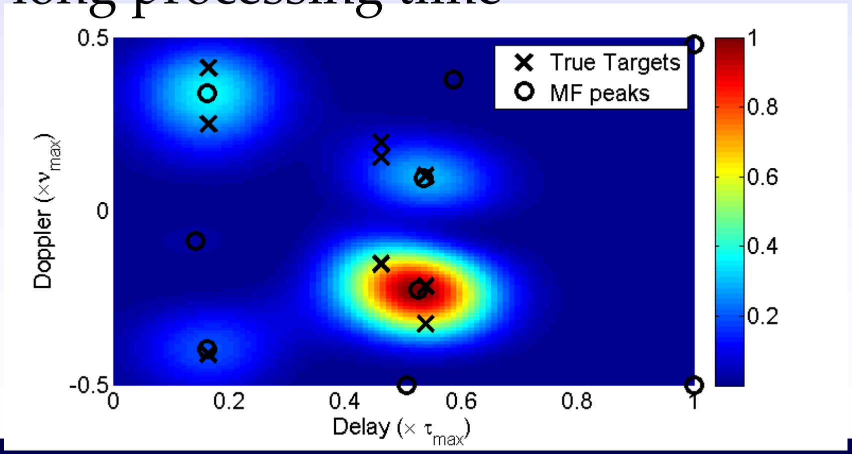
Compressed Beamforming

Resolution (1): Radar

- Principle:
  - A known pulse is transmitted
  - Reflections from targets are received
  - Target's ranges and velocities are identified


- Challenges:
  - Targets can lie on an arbitrary grid
  - Process of digitizing
    → loss of resolution in range-velocity domain
  - Wideband radar requires high rate sampling and processing which also results in long processing time

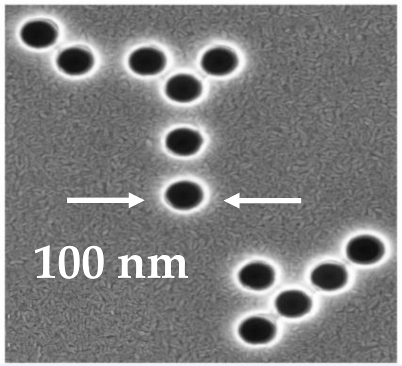


Resolution (2): Subwavelength Imaging

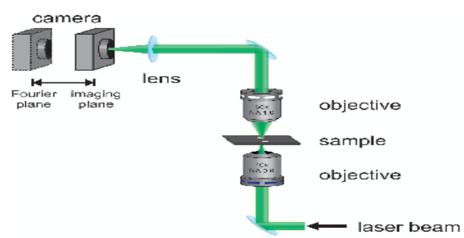
(Collaboration with the groups of Segev and Cohen)

Diffraction limit: Even a perfect optical imaging system has a resolution limit determined by the wavelength λ

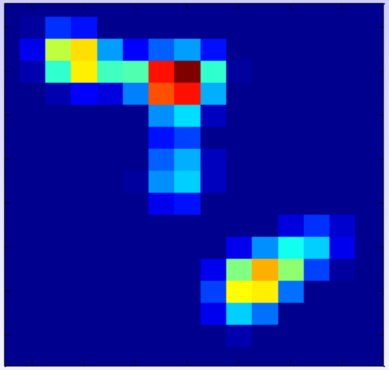
- The smallest observable detail is larger than ~ $\lambda/2$
- This results in image smearing



Nano-holes as seen in electronic microscope



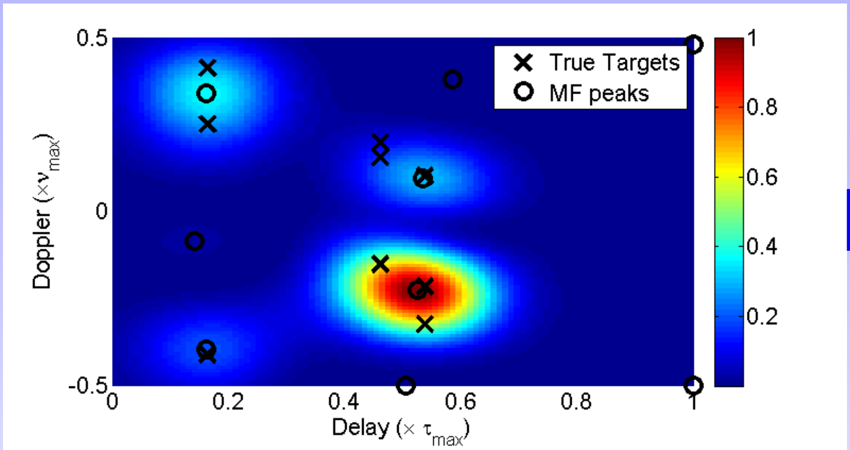
Sketch of an optical microscope: the physics of EM waves acts as an ideal low-pass filter

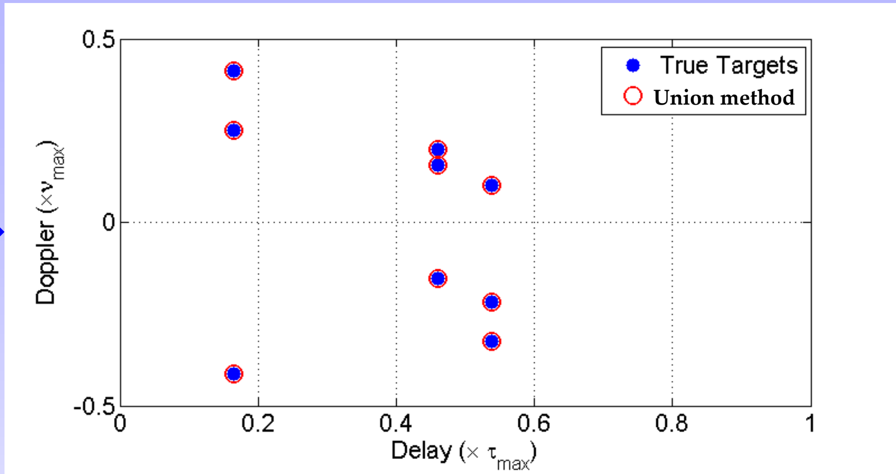


Blurred image seen in optical microscope

Imaging via "Sparse" Modeling

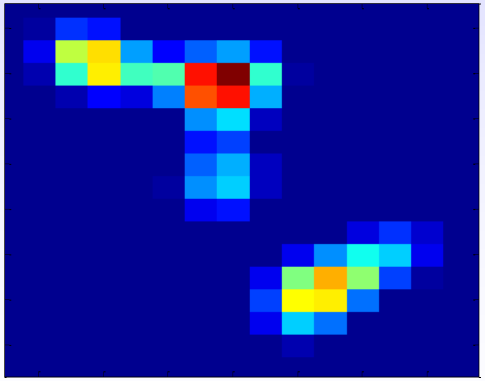
- Radar:

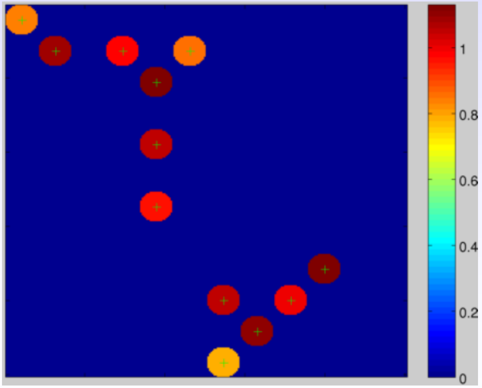




- Subwavelength Coherent Diffractive Imaging:

Bajwa *et al.*, '11





Recovery of sub-wavelength images from highly truncated Fourier power spectrum

Szameit *et al.*, Nature Photonics, '12

Proposed Framework

- Instead of a single subspace modeling use **union of subspaces** framework

- Adopt a new design methodology – **Xampling**
  - Compression+Sampling = Xampling
  - X prefix for compression, e.g. DivX

- Results in simple hardware and low computational cost on the DSP

Union + Xampling = Practical Low Rate Sampling

Talk Outline

- Motivation
- Classes of structured analog signals
- Xampling: Compression + sampling
- Sub-Nyquist solutions
  - Multiband communication: Cognitive radio
  - Time delay estimation: Ultrasound, radar, multipath medium identification
- Ultrasound and compressed beamforming
- Nonlinear compressed sensing: Phase retrieval

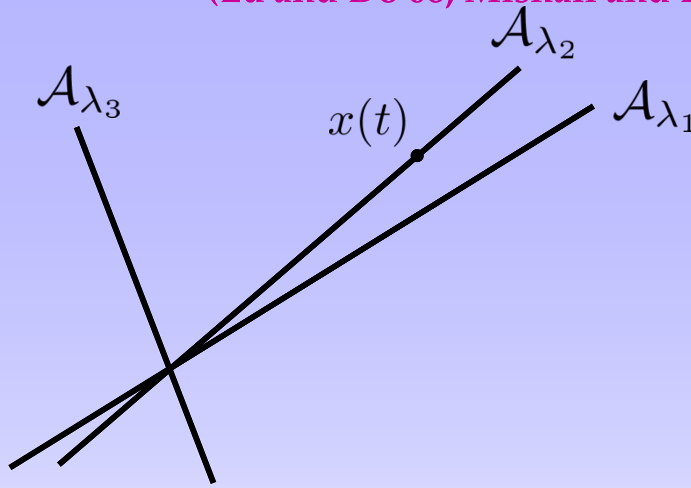
Union of Subspaces

(Lu and Do 08, Mishali and Eldar 09)

- Model: $\mathcal{U} = \bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda$

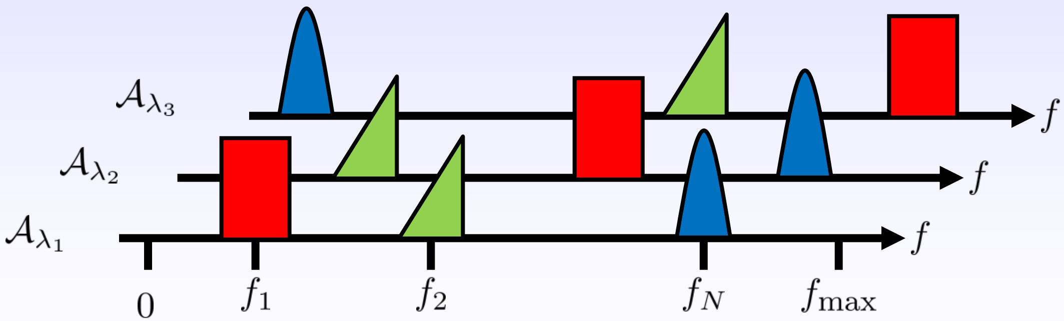
  $x(t) \in \mathcal{A}_{\lambda^*} \rightarrow \lambda^*$ is unknown a-priori

  Each $\mathcal{A}_\lambda$ has low dimension



- Examples:

Multiband communication

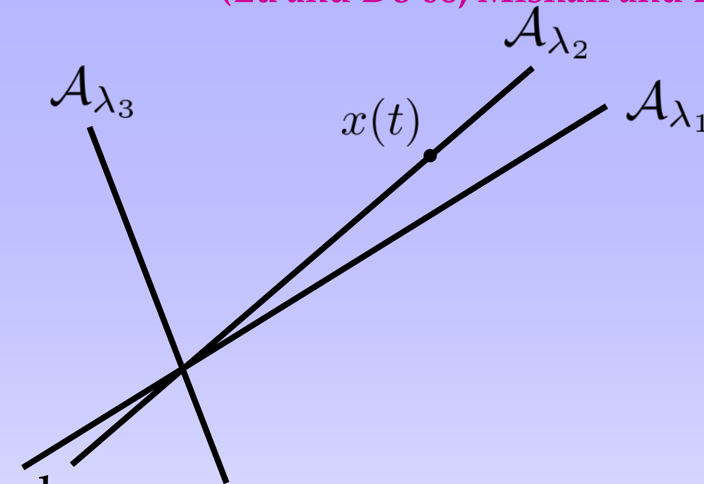


Union over possible band positions $f_i \in [0, f_{\max}]$

Union of Subspaces

(Lu and Do 08, Mishali and Eldar 09)

- Model: $\mathcal{U} = \bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda$

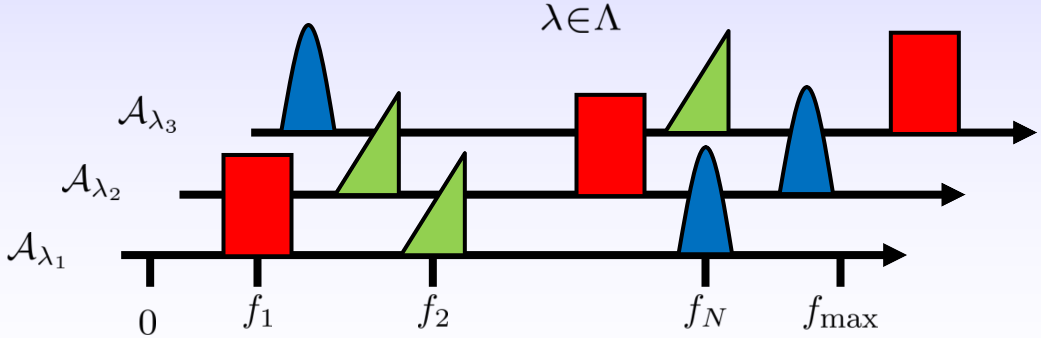


$x(t) \in \mathcal{A}_{\lambda^*}$ $\rightarrow$ λ^* is unknown a-priori

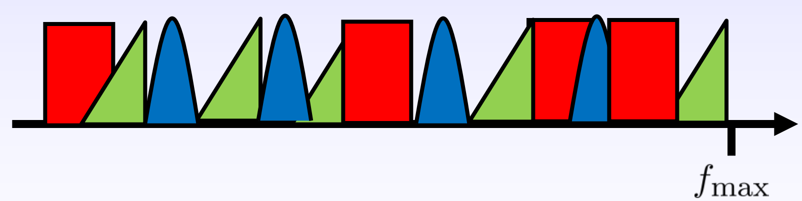
Each $\mathcal{A}_\lambda$ has low dimension

- Standard approach: Look at sum of all subspaces

$$\mathcal{U} = \bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda$$



$$\mathcal{U} = \oplus_{\lambda \in \Lambda} \mathcal{A}_\lambda$$



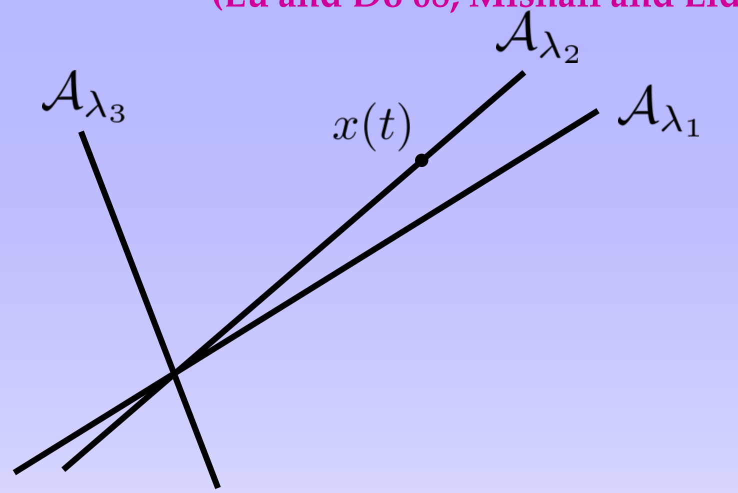
Signal bandlimited to $f_{\max}$

$\longrightarrow$ High rate

Union of Subspaces

(Lu and Do 08, Mishali and Eldar 09)

- Model: $\mathcal{U} = \bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda$



$x(t) \in \mathcal{A}_{\lambda^*}$ $\rightarrow$ λ^* is unknown a-priori

Each $\mathcal{A}_\lambda$ has low dimension

- Allows to keep low dimension in the problem model
- Low dimension translates to low sampling rate

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- Sub-Nyquist solutions
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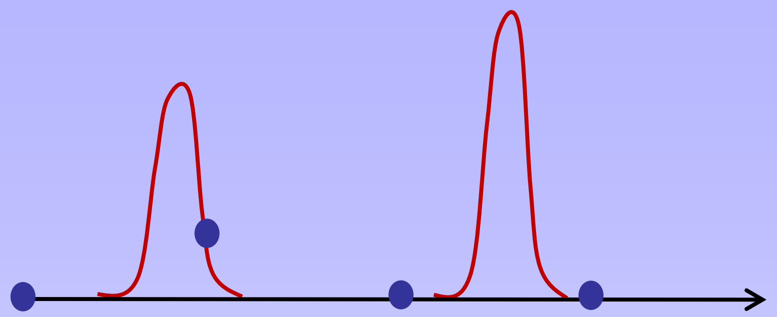
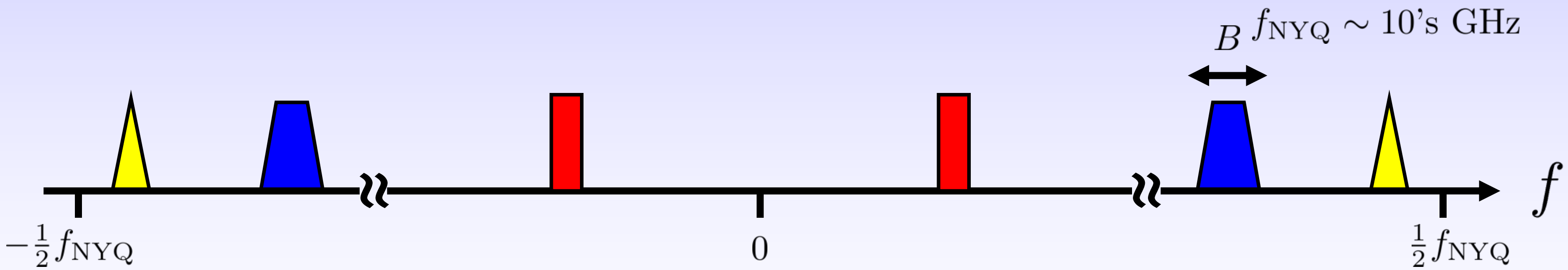
Difficulty

- Naïve attempt: direct sampling at low rate
- Most samples do not contain information!!

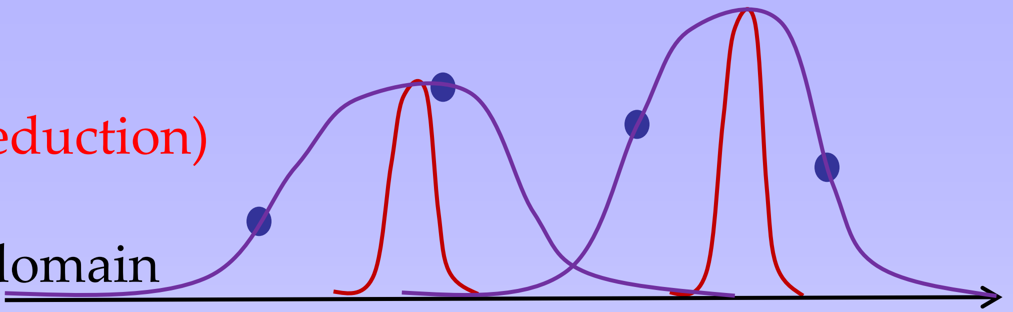


- Most bands do not have energy – which band should be sampled?

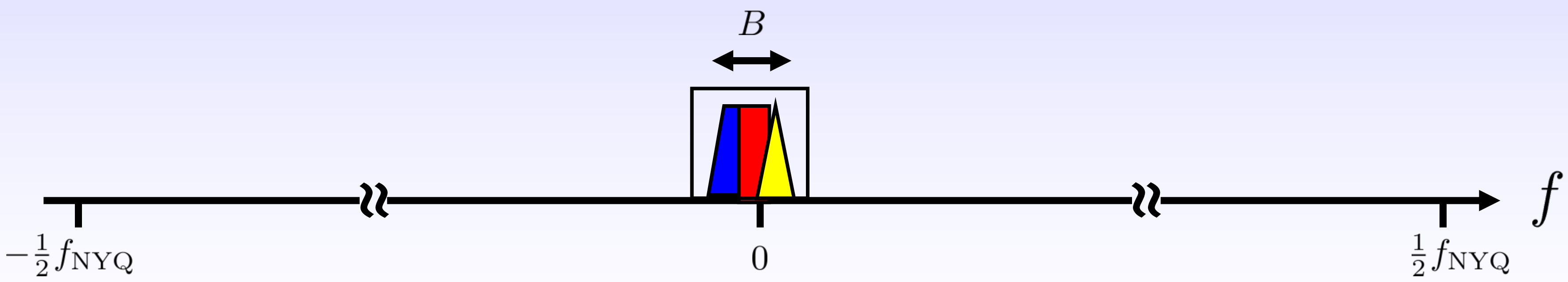


Intuitive Solution: Pre-Processing

- Smear pulse **before** sampling
  (analog projection – bandwidth reduction)
- Each sample contains energy
- Resolve ambiguity in the digital domain



- Alias all energy to baseband **before** sampling (analog projection)
- Can sample at low rate
- Resolve ambiguity in the digital domain



Xampling: Main Idea

- Create several streams of data
- Each stream is sampled at a low rate
  (overall rate much smaller than the Nyquist rate)
- Each stream contains a combination from different subspaces

Hardware design ideas

- Identify subspaces involved
- Recover using standard sampling results

DSP algorithms

Subspace Identification

For linear methods:

- Subspace techniques developed in the context of array processing (such as MUSIC, ESPRIT etc.)
- Compressed sensing: only for subspace identification

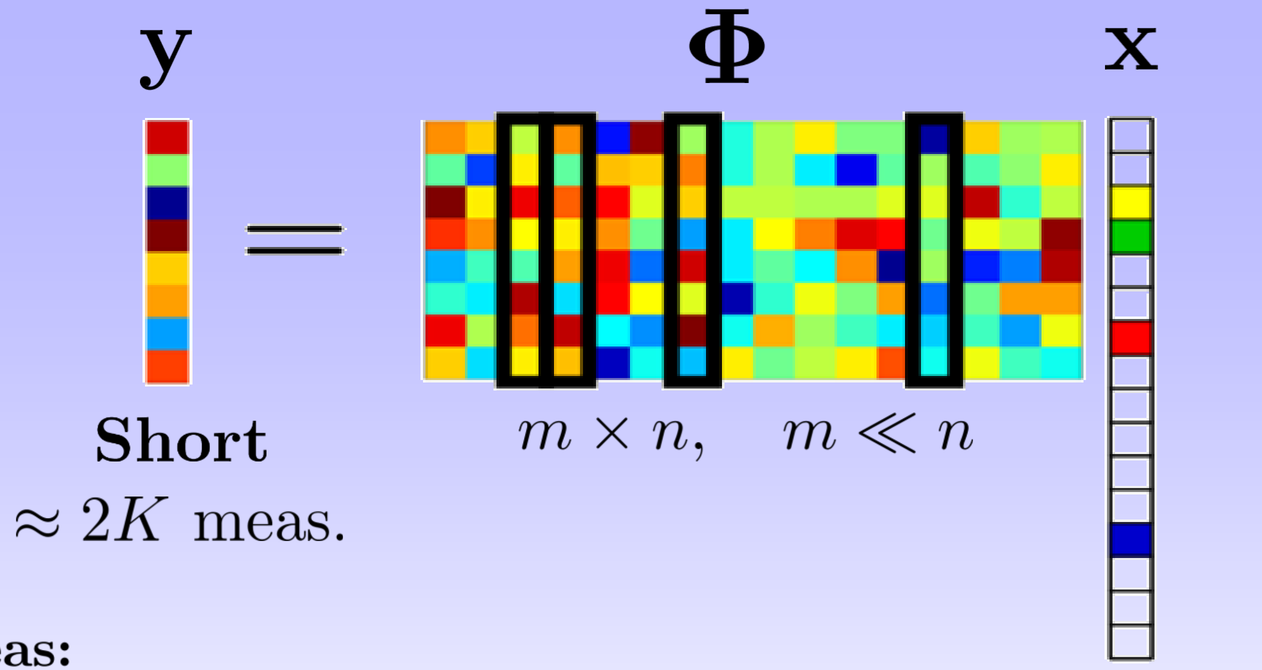
Connections between CS and subspace methods: (Malioutov, Cetin, and Willsky , Lee and Bresler, Davies and Eldar, Kim, Lee and Ye, Fannjiang, Austin, Moses, Ash and Ertin)

For nonlinear sampling:

- Specialized iterative algorithms: quadratic compressed sensing and more generally nonlinear compressed sensing

We use CS only after sampling and only to detect the subspace
Enables efficient hardware and low processing rates

Compressed Sensing



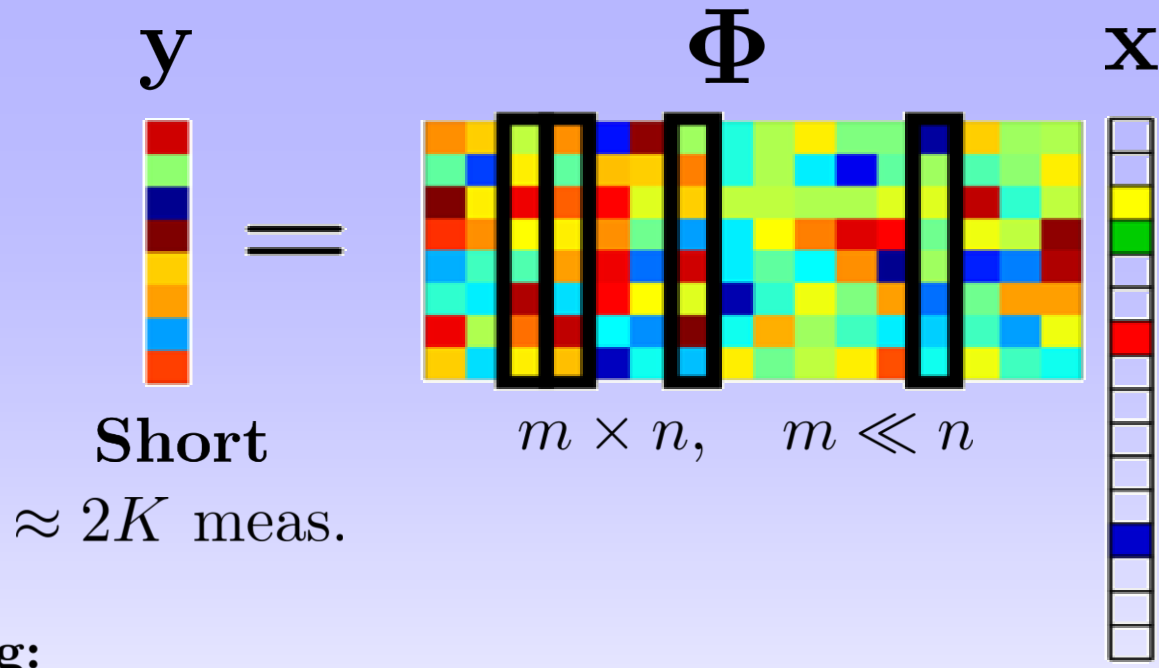
Main ideas:

- Sparse input vector with unknown support
- Sensing by sufficiently incoherent matrix (semi-random)
- Polynomial-time recovery algorithms

(*Candès, Romberg, Tao 2006*)

(*Donoho 2006*)

Compressed Sensing

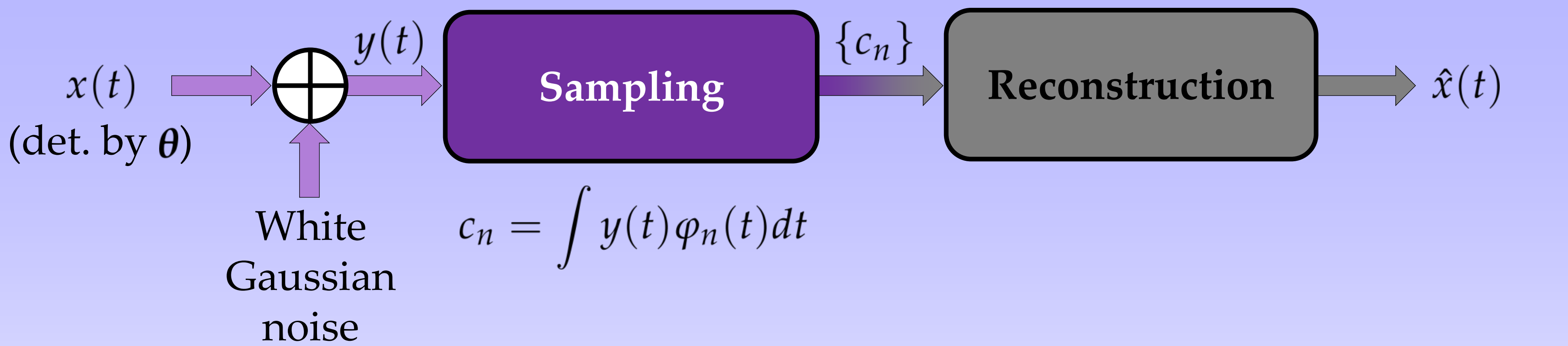


Xampling:

- Sparsity of $\boldsymbol{x}$ represents that only a few subspaces participate
- The matrix $\mathbf{\Phi}$ represents the aliasing of the hardware
- Support detection is equivalent to subspace detection

Optimal Xampling Hardware

(Ben-Haim, Michaeli and Eldar 10)



We derive two lower bounds on the performance of UoS estimation:

- Fundamental limit – regardless of sampling technique or rate
- Lower bound for a given sampling rate
  - Allows to determine optimal sampling method
  - Can compare practical algorithms to bound

Bounds for Noisy UoS

Theorem: Sample-Free CRB

Rate of innovation

Any unbiased estimator of $x(t)$ satisfies $\frac{1}{\tau}\text{MSE} \geq \rho_\tau \sigma^2$, regardless of the sampling method.

Rate of innovation: Number of degrees of freedom per unit time, coined by Vetterli et. al.

Bound on estimating a continuous time function:

- Typically bounds are derived for finite-dimensional parameters
- Here we need bounds on continuous-time ***structured*** functions
- To prove the bound we use ideas of CRB with measure theory and Pettis expectation

Bounds for Noisy UoS

Theorem: Sample-Free CRB

Rate of innovation

Any unbiased estimator of $x(t)$ satisfies $\frac{1}{\tau}\text{MSE} \geq \rho_\tau \sigma^2$, regardless of the sampling method.

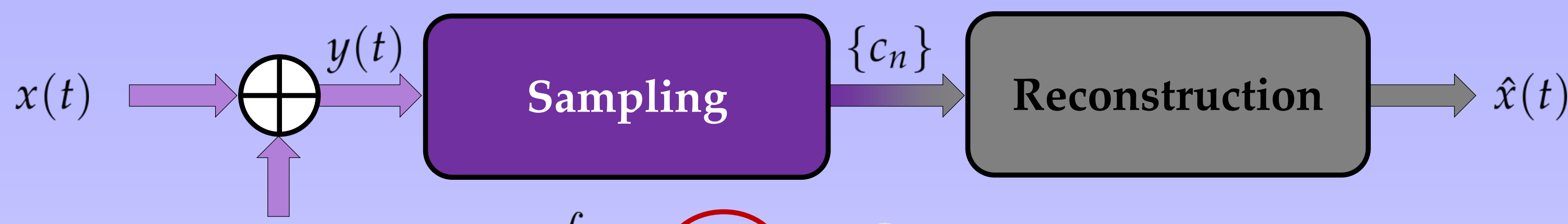
Theorem: CRB for given sampling method

Let $\hat{x}(t)$ be an unbiased estimator of a length-τ segment of $x(t)$ from samples $\{c[n] = \int \varphi_n(t) y(t) dt\}$. Then

$$\frac{1}{\tau}\text{MSE} \geq \frac{\sigma^2}{\tau} \text{Tr}\left\{ \left(\frac{\partial x}{\partial \boldsymbol{\theta}}\right)^* \left(\frac{\partial x}{\partial \boldsymbol{\theta}}\right) \left[\left(\frac{\partial x}{\partial \boldsymbol{\theta}}\right)^* \boldsymbol{P}_\Phi \left(\frac{\partial x}{\partial \boldsymbol{\theta}}\right) \right]^{-1} \right\}$$

where $\boldsymbol{\theta}$ are the parameters defining $x(t)$ in the given segment and $\boldsymbol{P}_\Phi$ is the orthogonal projector onto the subspace Φ spanned by the sampling kernels $\{\varphi_n(t)\}$.

Optimal Sampling



$$c_n = \int y(t)\varphi_n(t)dt \quad ?$$

- **Goal**: recover a segment of a random process $x(t)$, with autocorrelation $R_X(t,\eta) = \mathbb{E}[x(t)x(\eta)]$ from N samples
- **Method**: optimize MSE using previous bound

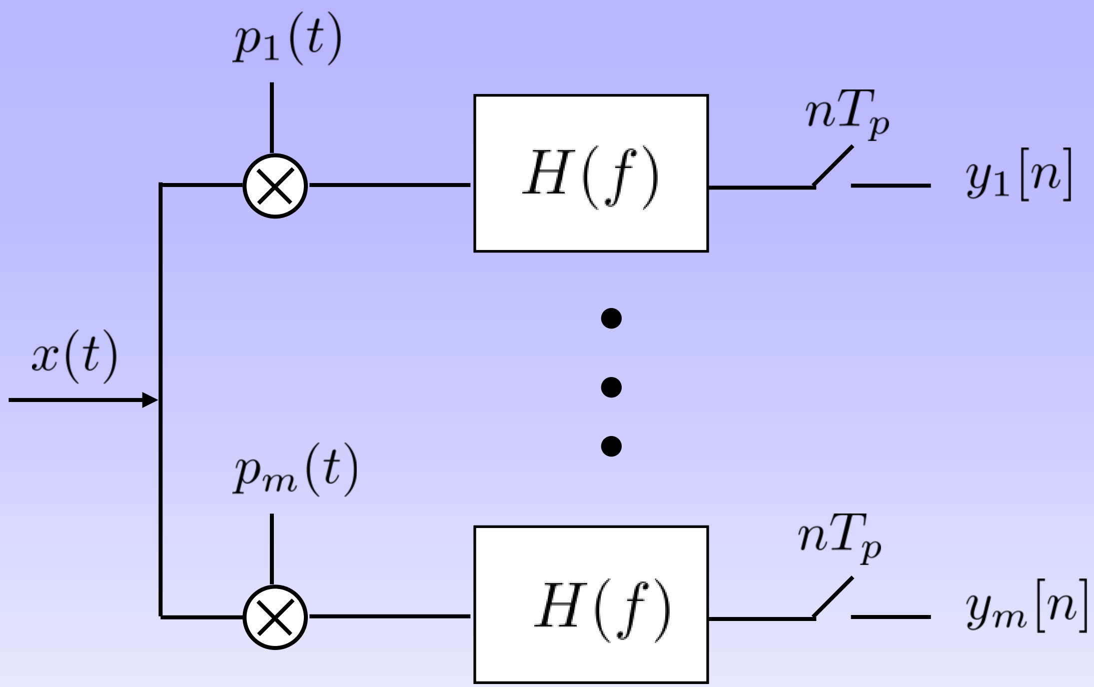
Theorem (Generalized KLT)

The minimal MSE is obtained with $\varphi_n(t) = \psi_n(t)$ where $\psi_n(t)$ are the eigenfunctions of R_X

- When $R_X(t,\eta) = R_X((t-\eta) \mod T)$ then $\psi_n(t) = \frac{1}{\sqrt{T}}e^{j\frac{2\pi}{T}nt}$

Sampling with Sinusoids is Optimal

Xampling Hardware



- $p_i(t)$ - periodic functions
- $p_i(t) = \sum a_{in} e^{-j\frac{2\pi}{T_p}nt}$ sums of exponentials
- The filter *H(f)* allows for additional freedom in shaping the tones
- The channels can be collapsed to a single channel

Some Earlier Work …

- Prony 1795, Caratheodory 1900, Rife and Boorstyn 70s: Sampling of pure tones
- Beurling 1938: Spectrum extrapolation of pulses using CT L1
- Bresler, Feng, and Venkataramani 1996-2000: Certain classes of MB signals
- Vetterli et. al. 2002: Finite rate of innovation framework
- Tropp et. al. 2010: Random demodulator

Goal: Target System-Level Challenges

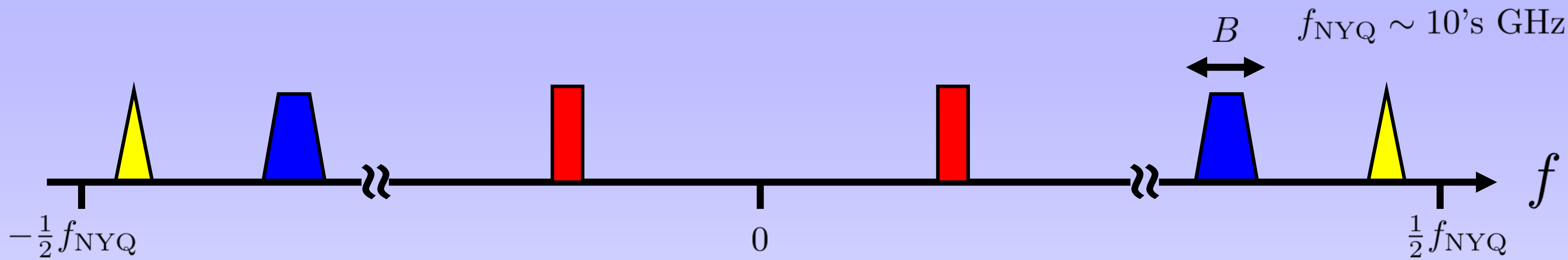
- Unified framework for continuous time models
- Broad class of signals
- Efficient and robust hardware
- Low rate DSP
- Applications: nonlinearities, correlations and more

Talk Outline

- Motivation
- Classes of structured analog signals
- Xampling: Compression + sampling
- Sub-Nyquist solutions
  - Multiband communication
  - Time delay estimation: Ultrasound, radar, multipath medium identification
- Ultrasound and compressed beamforming
- Nonlinear compressed sensing: Phase retrieval

Signal Model

(Mishali and Eldar 07-09)



1. Each band has an uncountable number of non-zero elements
2. Band locations lie on the continuum
3. Band locations are unknown in advance

$$\mathcal{M} = \left\{ x(t) \,\middle|\, \text{no more than } N \text{ bands, max width } B\text{, bandlimited to } [-\tfrac{1}{2} f_{\text{NYQ}}, +\tfrac{1}{2} f_{\text{NYQ}}) \right\}$$

Rate Requirement

Theorem (Single multiband subspace)

Let R be a sampling set for $\mathcal{B}_{\mathcal{F}} = \{x(t) \in L^2(\mathbb{R}) \,|\, \operatorname{supp} X(f) \subseteq \mathcal{F}\}$. Then,

$$\underbrace{D^-(R)}_{\text{Average sampling rate}} \geq \lambda = |\mathcal{F}|$$

(Landau 1967)

Theorem (Union of multiband subspaces)

Let R be a sampling set for $\mathcal{N}_\lambda = \bigcup_{|\mathcal{F}| \leq \lambda} \mathcal{B}_{\mathcal{F}}$.

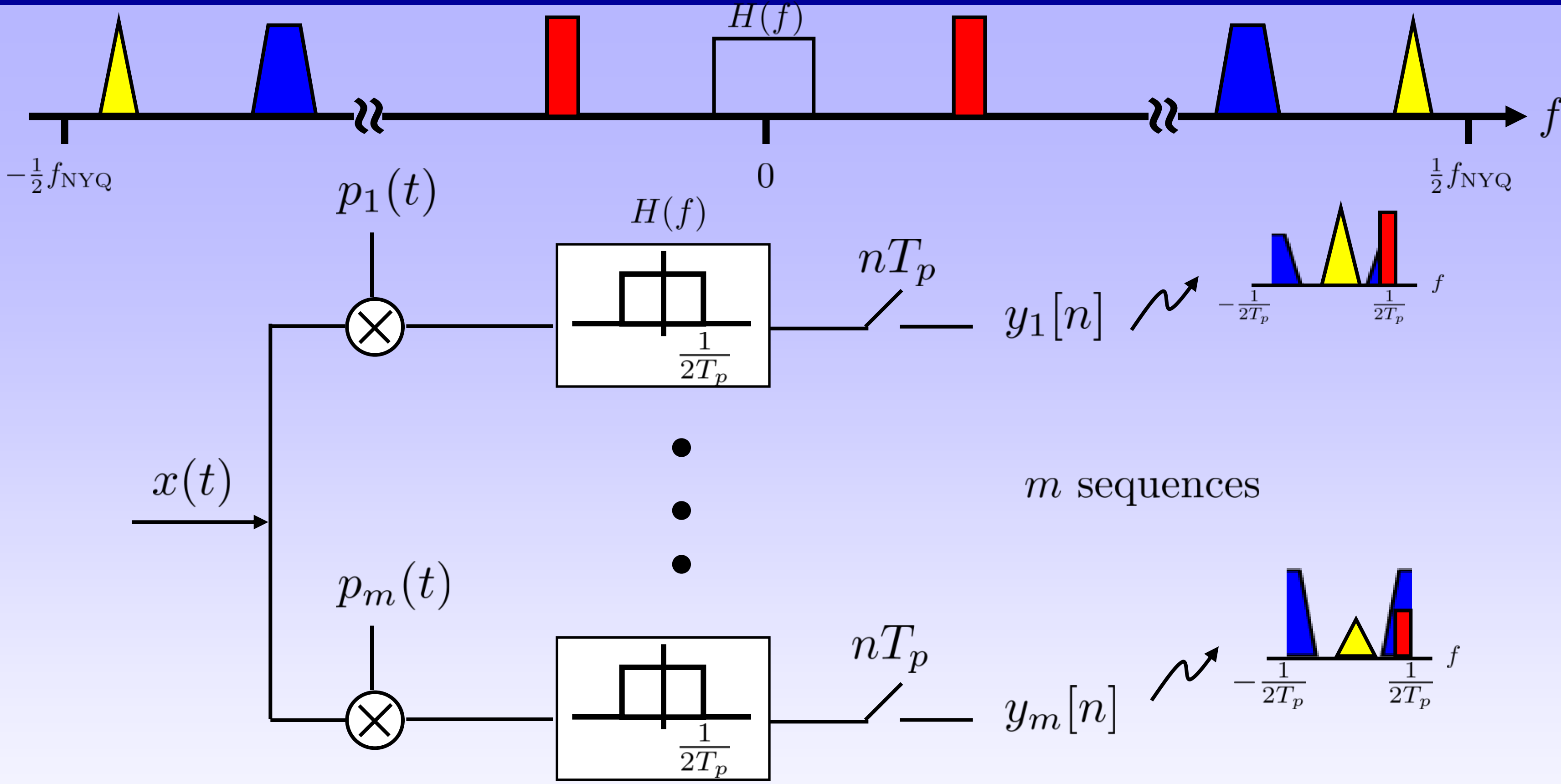
Then,

$$D^-(R) \geq \min\{2\lambda, f_{\mathrm{NYQ}}\}$$

(Mishali and Eldar 2007)

1. The minimal rate is doubled.
2. For $x(t) \in \mathcal{M}$, the rate requirement is $2NB$ samples/sec (on average).

The Modulated Wideband Converter



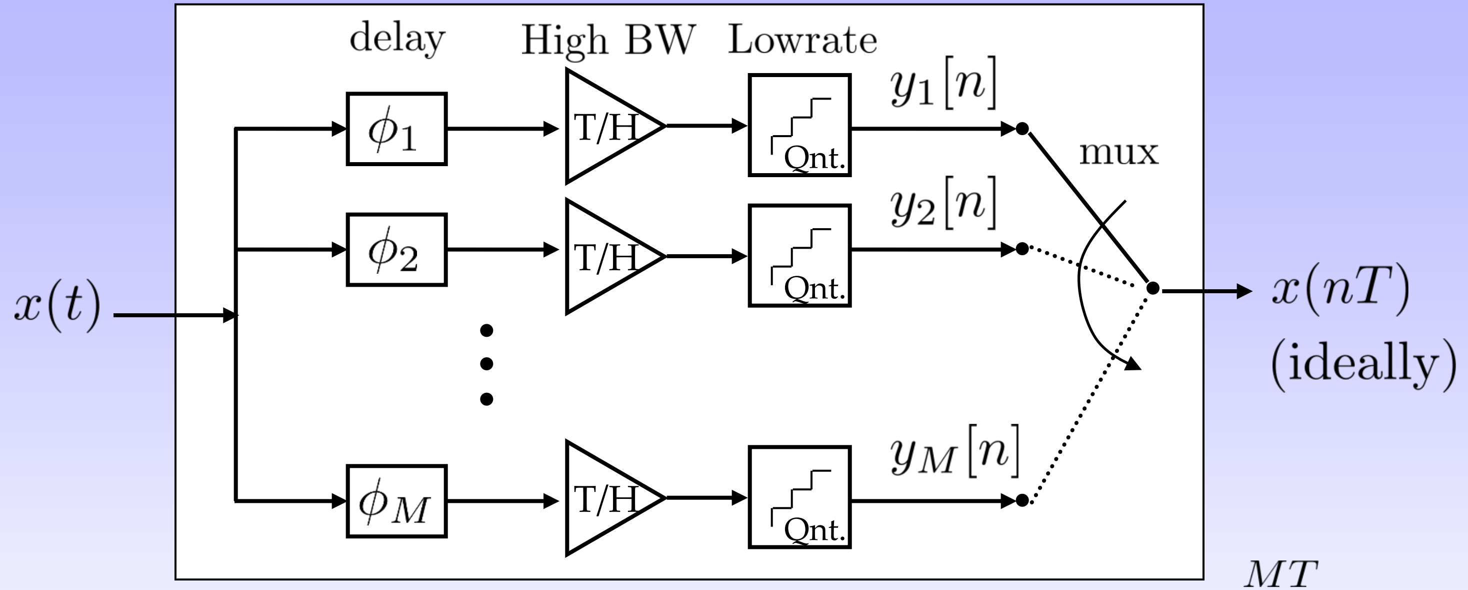
T_p−periodic $p_i(t)$ gives the desired aliasing effect

and many more...

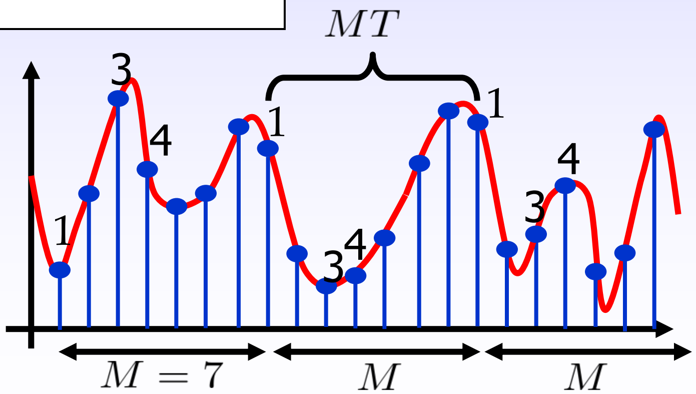
Time-Interleaved ADCs

(Lin and Vaidyanathan, Herley and Wong, Feng and Bresler, Mishali and Eldar)

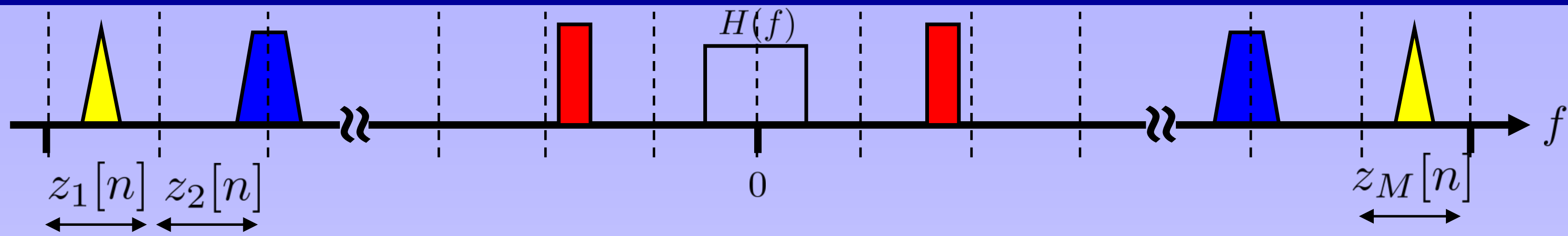
A high-rate ADC comprised of a bank of lowrate devices



- Both T/H and mux operate at the Nyquist rate
- Digital processing and recovery requires interpolation to the high Nyquist grid
- Accurate time-delays ϕ_i are needed
- Channels cannot generally be collapsed



Recovery From Xamples

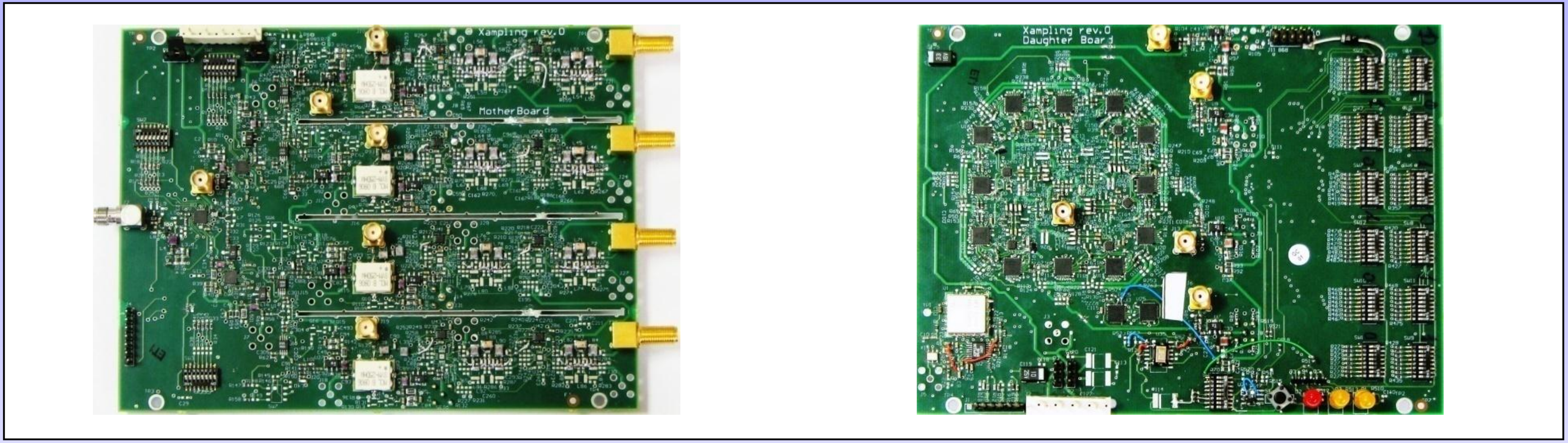


$$\begin{pmatrix} y_1[n] \\ \vdots \\ y_m[n] \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{il} = c_{il} \\ \\ m \times M \end{pmatrix} \begin{pmatrix} z_1[n] \\ z_2[n] \\ \vdots \\ \\ z_M[n] \end{pmatrix}$$

- Spectrum sparsity: Most of the $z_i[n]$ are identically zero
- For each *n* we have a small size CS problem
- Problem: CS algorithms for each *n* → many computations
- Solution: Use the ``CTF'' block which exploits the joint sparsity and reduces the problem to a single finite CS problem

A 2.4 GHz Prototype

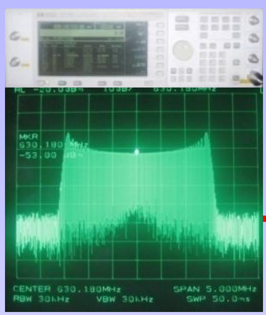
(Mishali, Eldar, Dounaevsky, and Shoshan, 2010)



- Rate proportional to the actual band occupancy
- All DSP done at low rate as well
- 2.3 GHz Nyquist-rate, 120 MHz occupancy
- 280 MHz sampling rate
- Wideband receiver mode:
  - 49 dB dynamic range, SNDR > 30 dB over all input range
- ADC mode:
  - 1.2 volt peak-to-peak full-scale, 42 dB SNDR = 6.7 ENOB

Sub-Nyquist Demonstration

Carrier frequencies are chosen to create overlayed aliasing at baeband

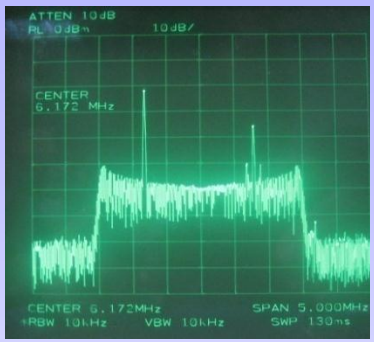


FM @ 631.2 MHz

AM @ 807.8 MHz

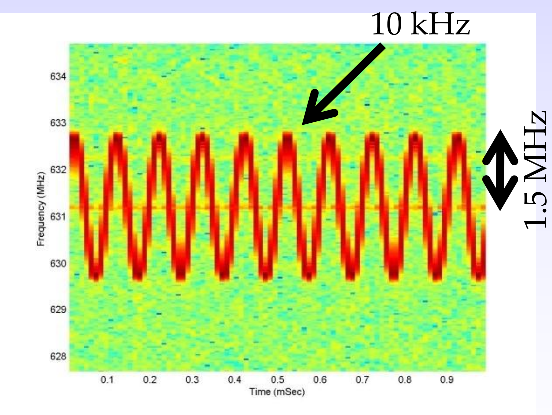
Sine @ 981.9 MHz

MWC prototype

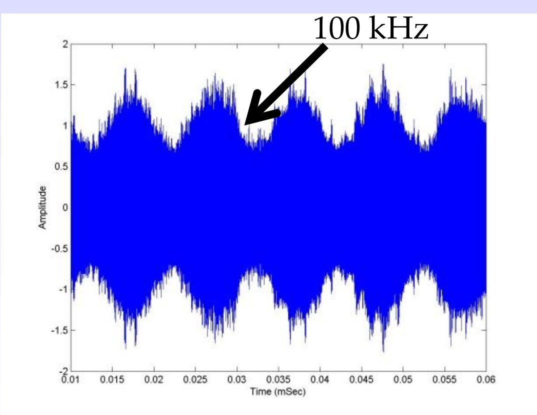


aliasing around 6.171 MHz

Reconstruction
(CTF)



FM @ 631.2 MHz

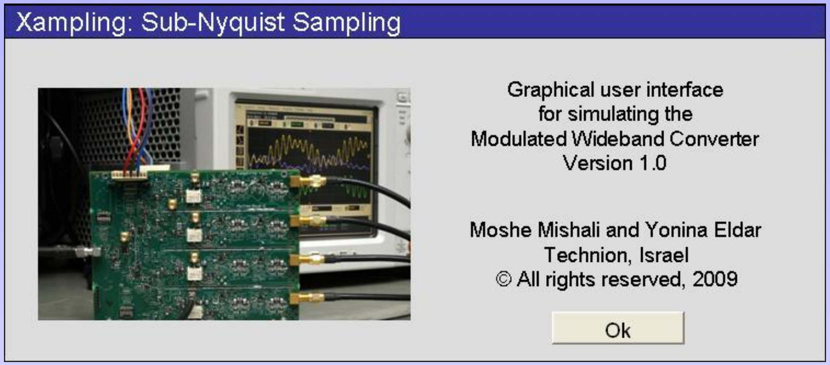


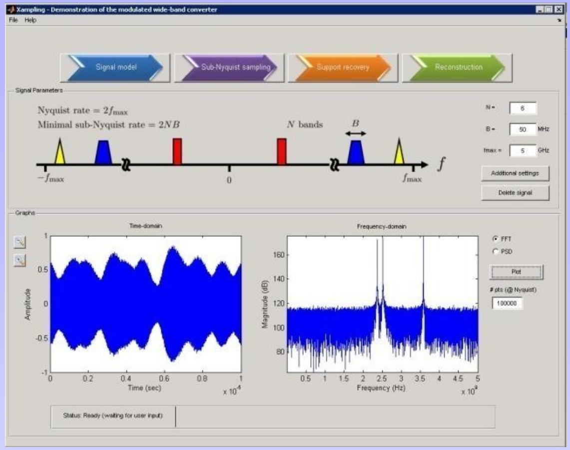
AM @ 807.8 MHz

Mishali et al., 10

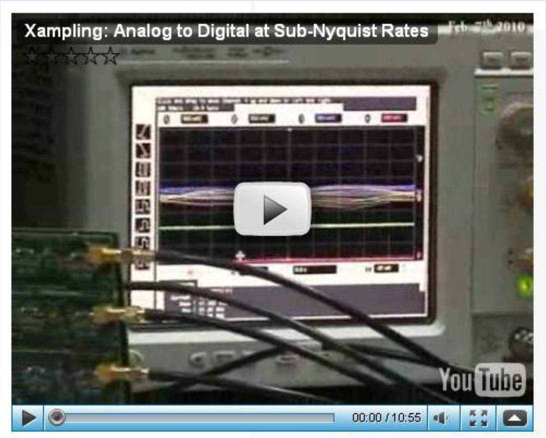
Online Demonstrations

- GUI package of the MWC



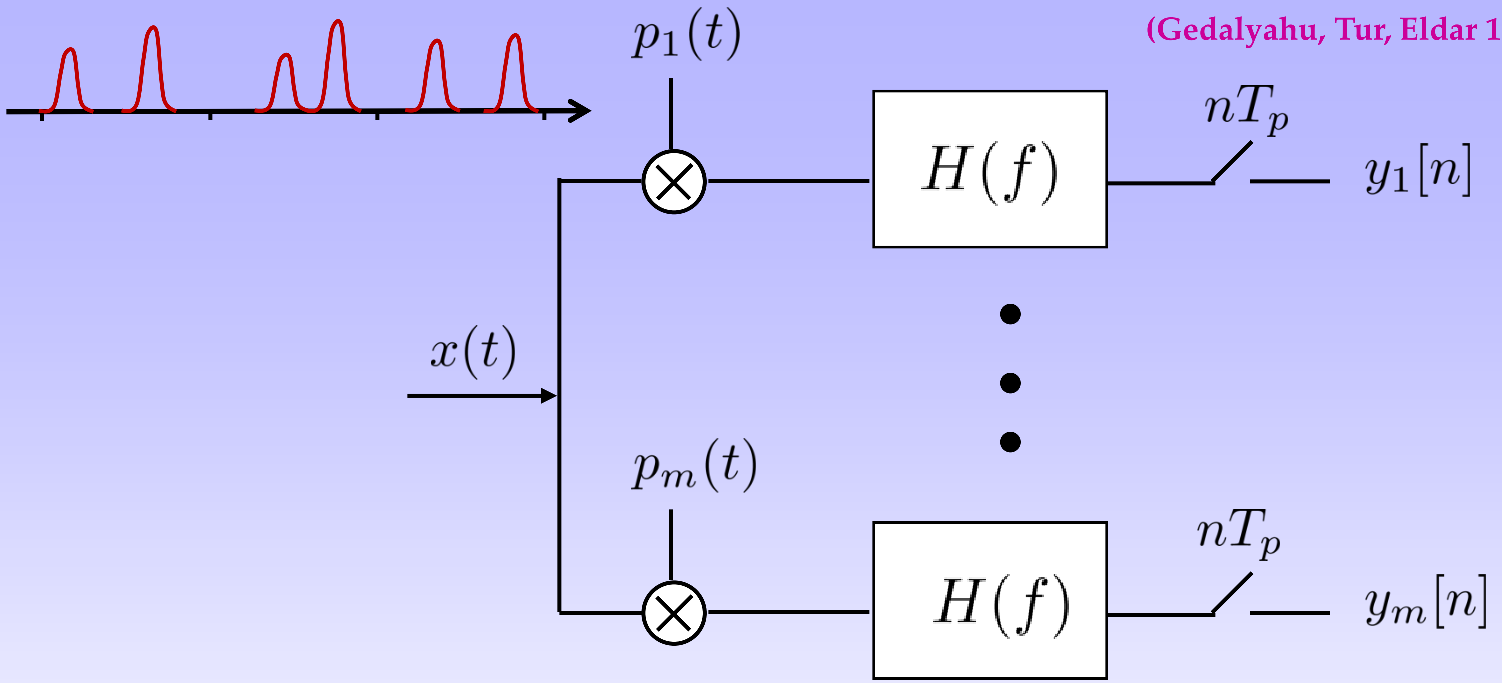


- Video recording of sub-Nyquist sampling + carrier recovery in lab

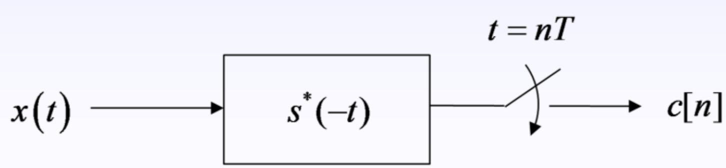


Streams of Pulses

(Gedalyahu, Tur, Eldar 10, Tur, Freidman, Eldar 10)



- $H(f)$ is replaced by an integrator
- Can equivalently be implemented as a single channel with $T = T_p/m$

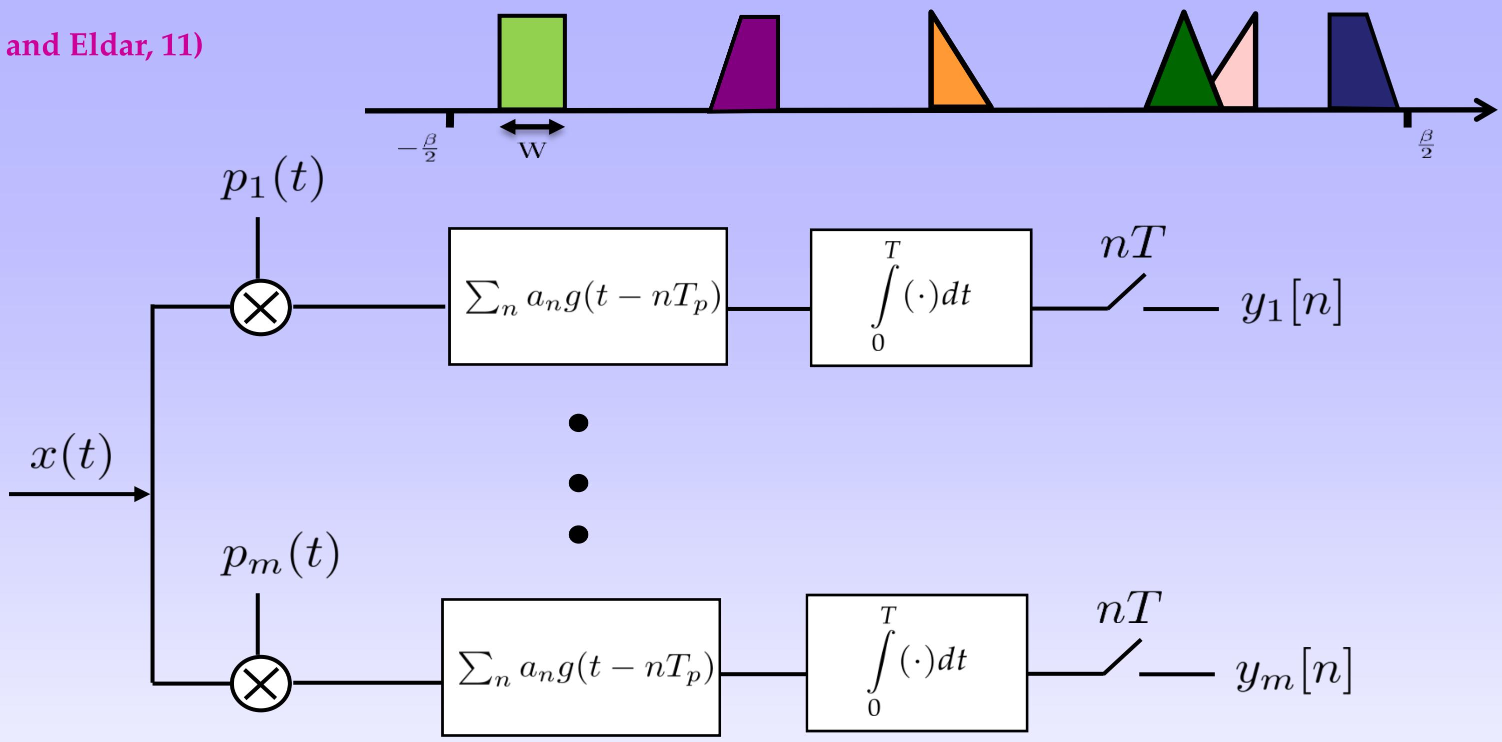


$$s(t) = \sum_n b_n e^{j\frac{2\pi}{T_p}nt}\mathrm{rect}(t/T_p)$$

- Application to radar, ultrasound and general localization problems such as GPS

Unknown Pulses

(Matusiak and Eldar, 11)

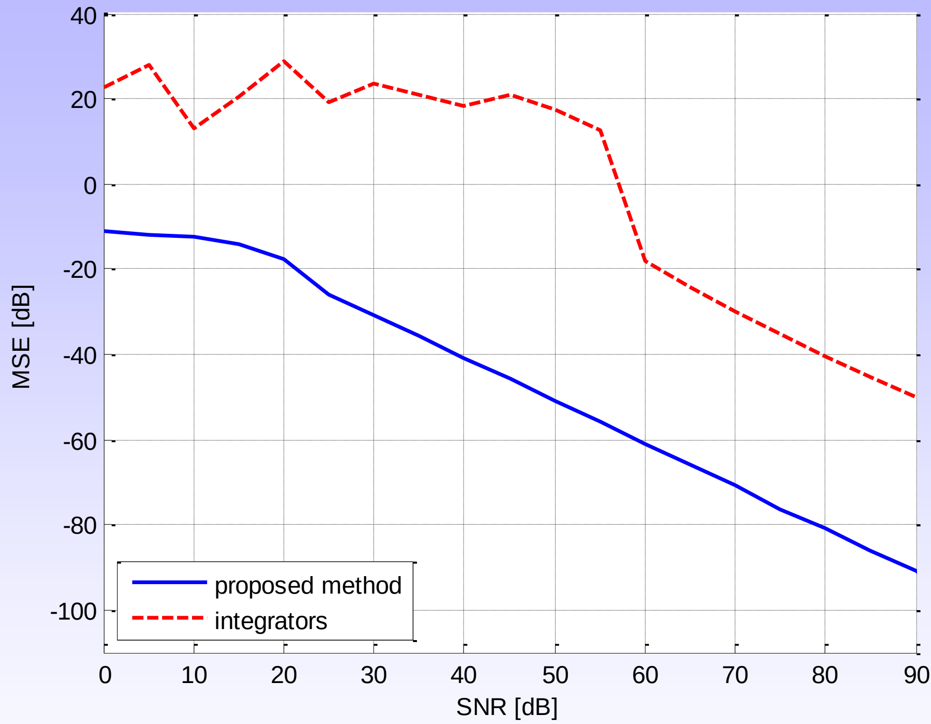


- Output corresponds to aliased version of Gabor coefficients
- Recovery by solving 2-step CS problem $Y = AZB^T$ Row-sparse Gabor Coeff.
  1. Solve $Y = AC$ with $C = ZB^T \Rightarrow$ Since Z is row-sparse C is row-sparse
  2. Solve CS problem $C^T = BZ$ where Z is row sparse

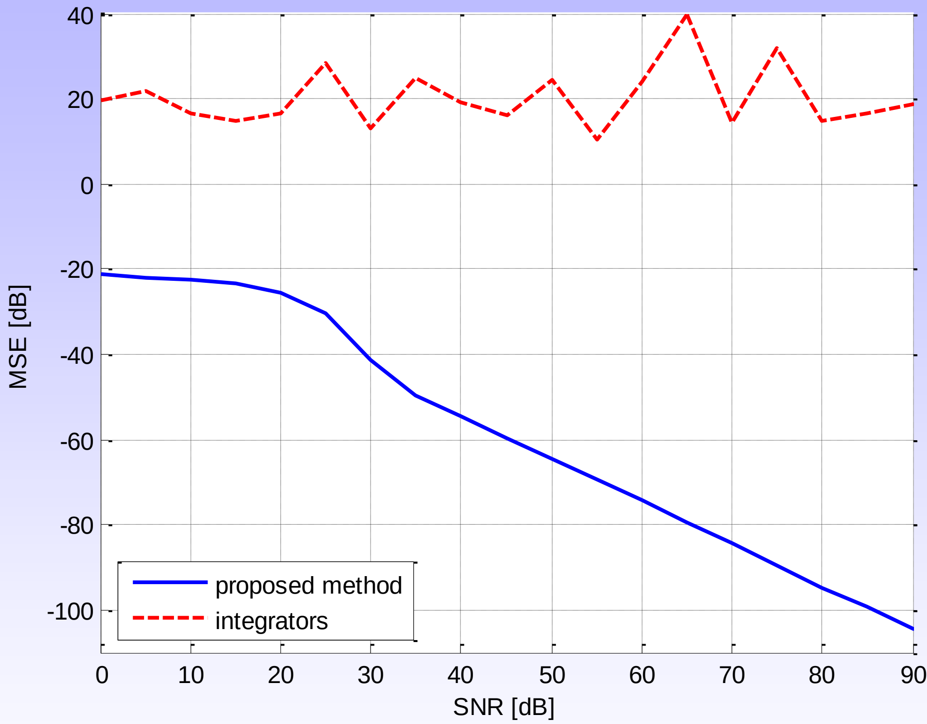
Noise Robustness

- MSE of the delays estimation, versus integrators approach *(Kusuma & Goyal)*

L=2 pulses, 5 samples



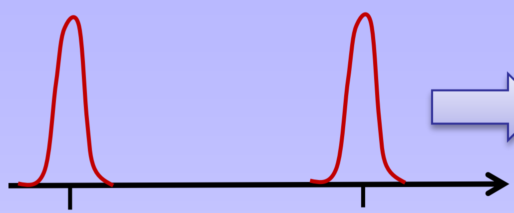
L=10 pulses, 21 samples



The proposed scheme is stable even for high rates of innovation!

Application:
Multipath Medium Identification

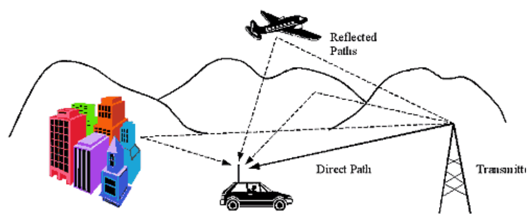
(Gedalyahu and Eldar 09-10)

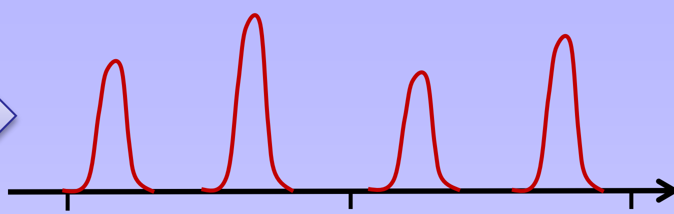


$$x_T(t) = \sum_{n \in \mathbb{Z}} h(t - nT)$$

LTV channel

L propagation paths





$$x_R(t) = \sum_{l \in \mathbb{Z}} a_l h(t - t_l)$$

L pulses per period T

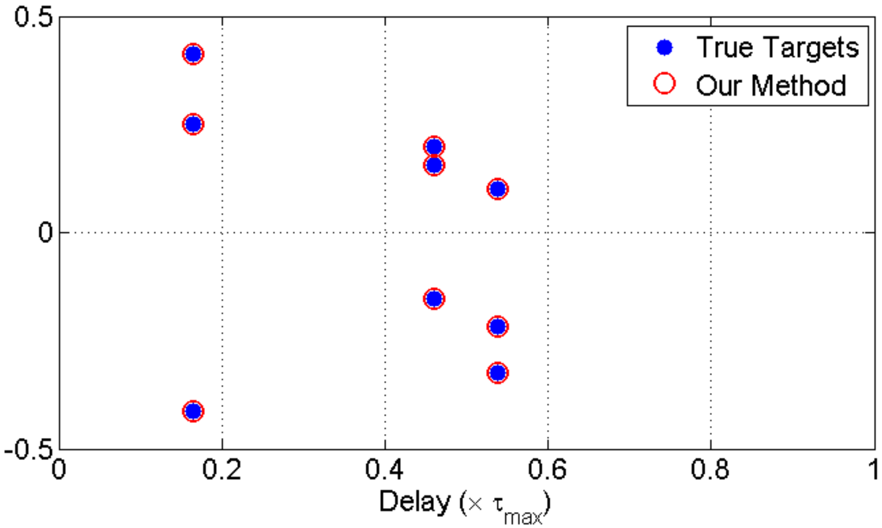
- Medium identification (collaboration with National Instruments):
  - Recovery of the time delays
  - Recovery of time-variant gain coefficients

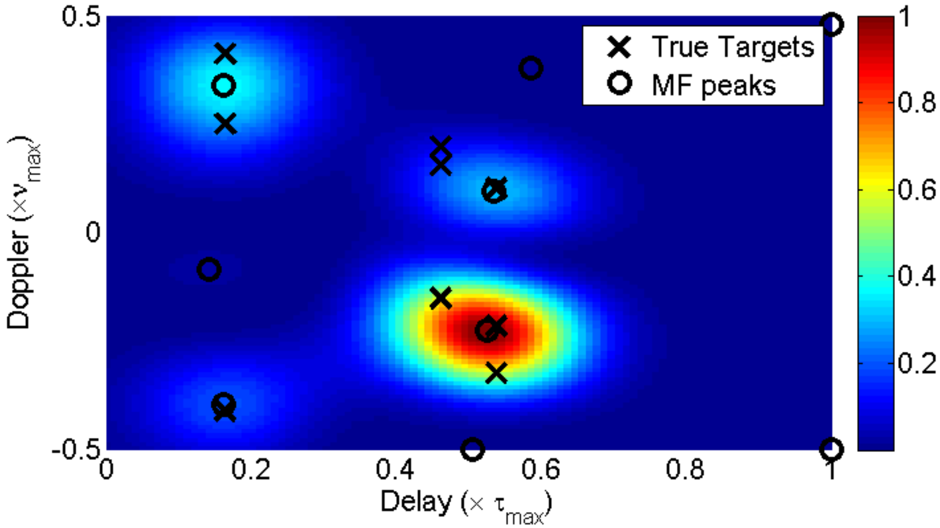
The proposed method can recover the channel parameters from sub-Nyquist samples

Application: Radar

(*Bajwa, Gedalyahu and Eldar, 10*)

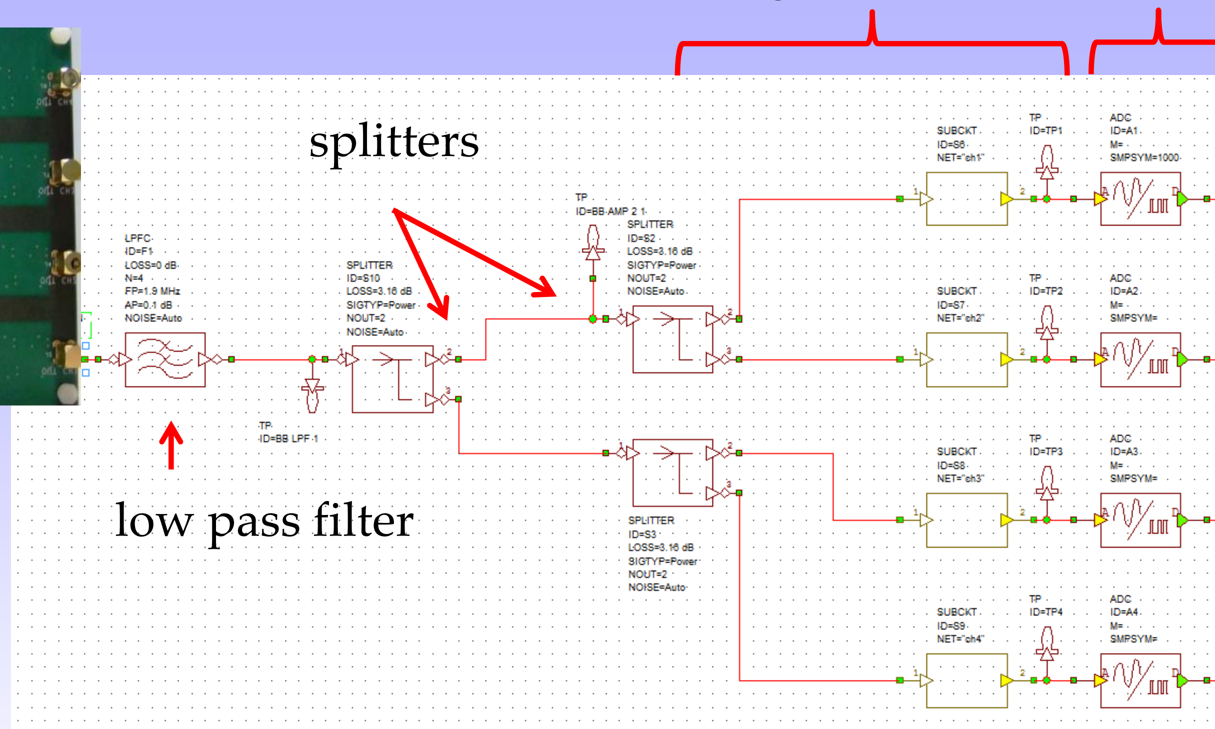
- Each target is defined by:
  - Range – delay
  - Velocity – doppler
- In theory, targets can be identified with **infinite resolution** as long as the time-bandwidth product satisfies $\mathcal{TW} \geq 2\pi(K+1)^2$
- Previous results required infinite time-bandwidth product

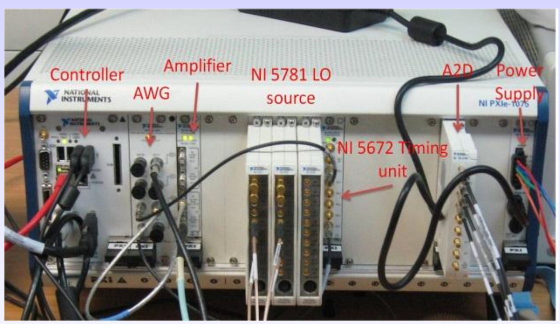


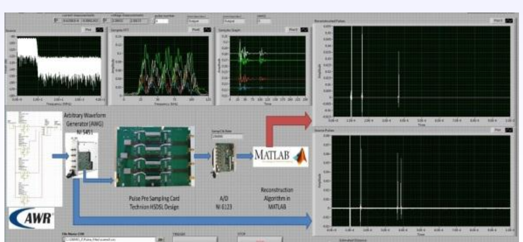


Xampling of Radar Pulses

(Itzhak et. al. 2012 in collaboration with NI)







Demo of real-time radar at NI week as we speak ..

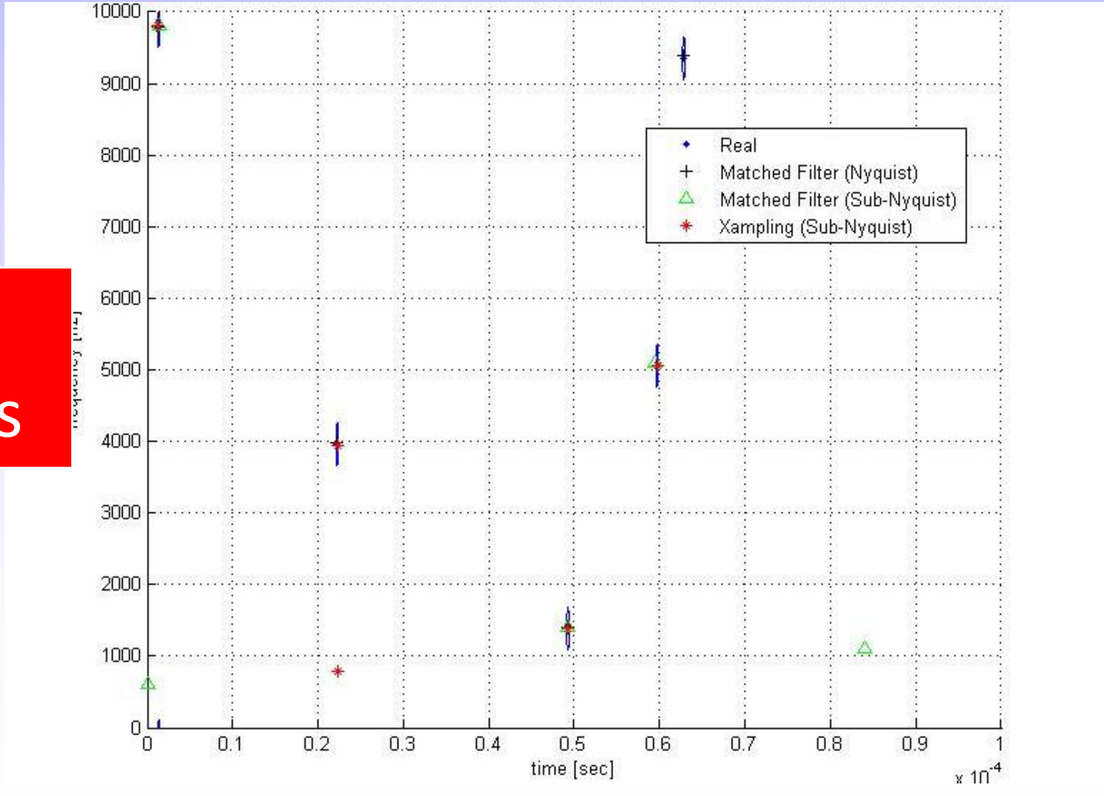
NATIONAL INSTRUMENTS™

Low SNR: -25 dB

(Bar-Ilan and Eldar, 12)

- Target delay and Doppler chosen randomly in the continuous unambiguous interval
- L =5, PRI = 0.1 mSec, P = 100 pulses, bandwidth B = 10MHz
- Nyquist rate generates 2000 samples / pulse, Xampling uses 200 samples

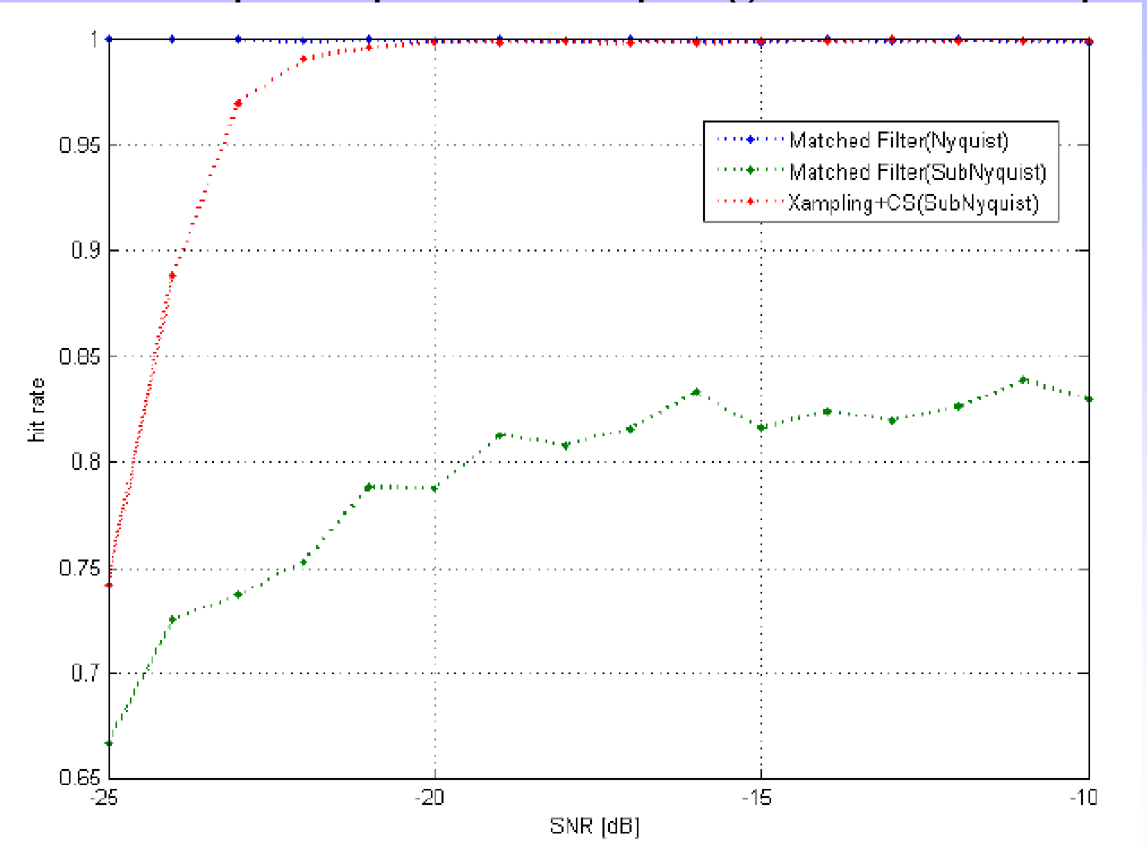
MF: 2/5 detections
Xampling: 4/5 detections



Low SNR

- Target delay and Doppler chosen randomly in the continuous unambiguous interval
- L = 5, PRI = 0.1 mSec, P = 100 pulses, bandwidth B = 10MHz
- Nyquist rate generates 2000 samples / pulse, Xampling uses 200 samples

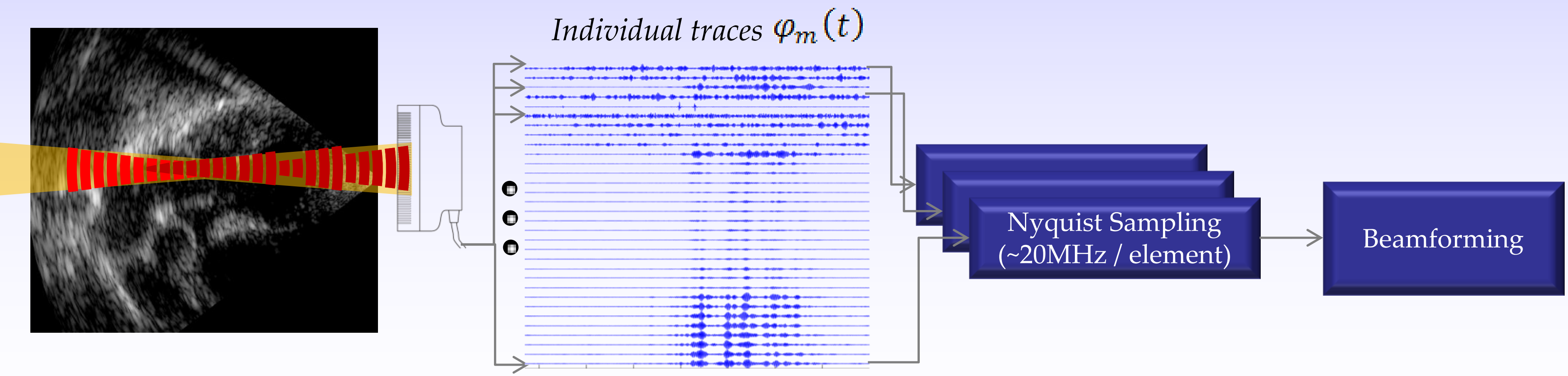
Hit rate as a function of SNR



Application to Ultrasound

Wagner, Eldar, and Friedman, '11

- Ultrasonic pulse is transmitted into the tissue
- Pulse is conducted along a relatively narrow beam
- Echoes are scattered by density and propagation-velocity perturbations
- Reflections detected by multiple array elements.
- Beamforming is applied – digital processing , signals must first be sampled at Nyquist rate (~20MHz)



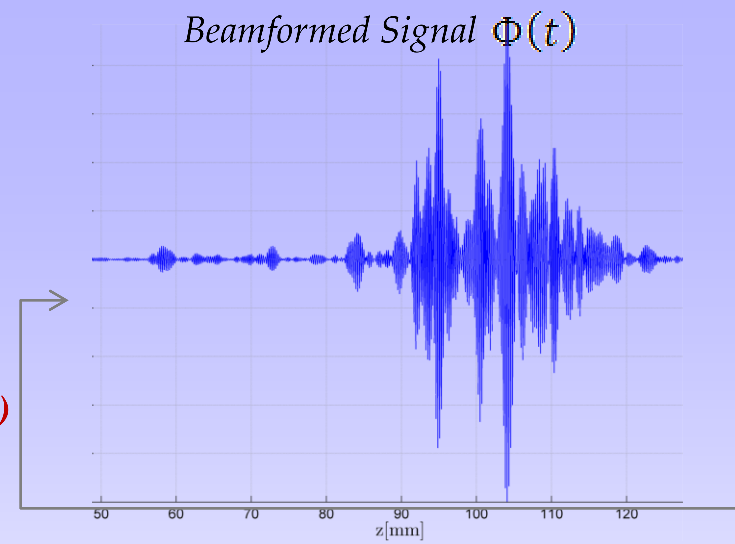
Standard Imaging - Beamforming

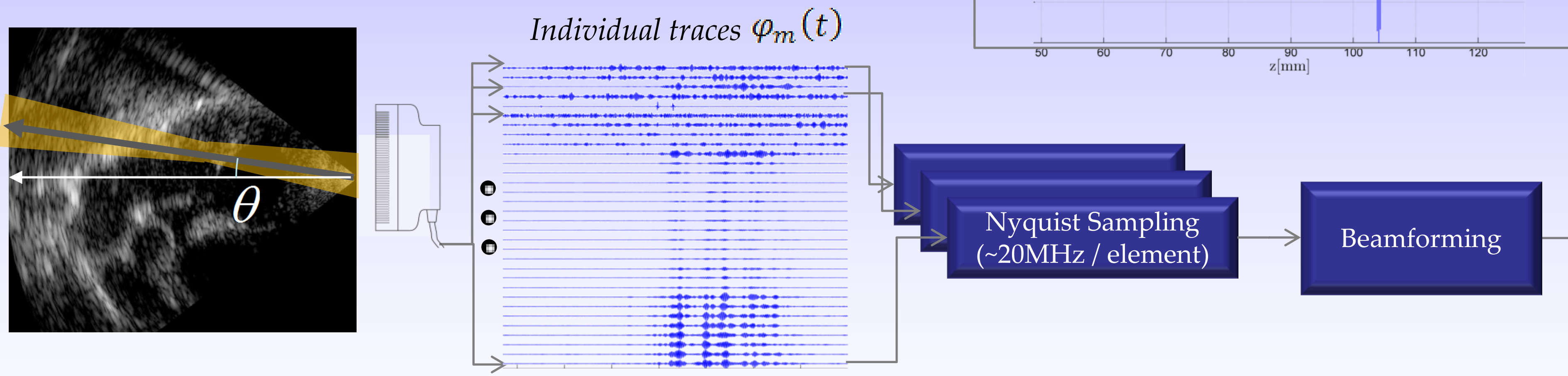
Non-linear scaling of the received signals

$$\Phi(t;\theta)=\frac{1}{M}\sum_{m=1}^{M}\varphi_m\left(\frac{1}{2}\left(t+\sqrt{t^2-4\gamma_m t\sin\theta+4\gamma_m^2}\right)\right)$$

γ_m - distance from m'th element to origin , normalized by c .

Performed in the digital domain (after sampling at Nyquist-rate)



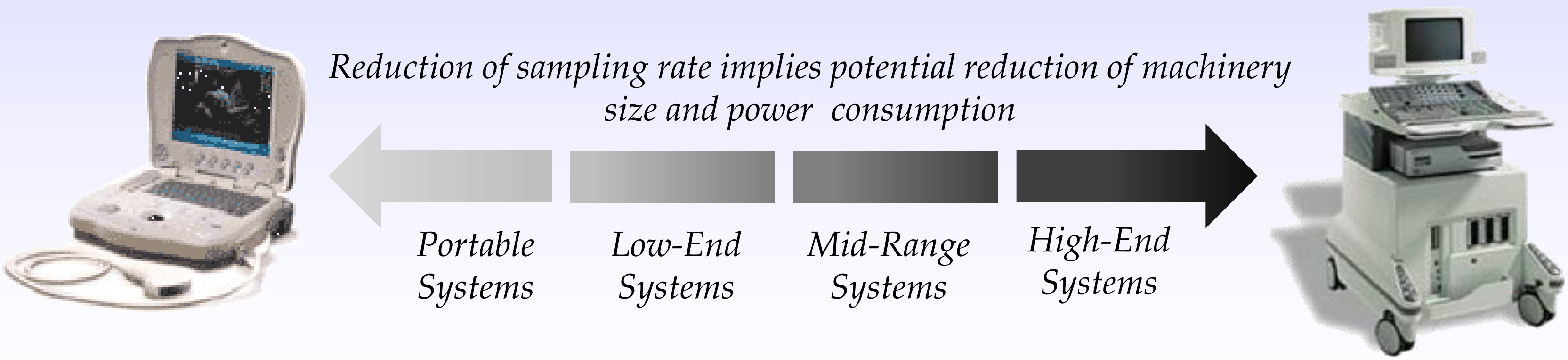


- **Focusing along a certain axis – reflections originating from off-axis are attenuated (destructive interference pattern)**
- **SNR is improved**

Sample Rate Reduction - Motivation

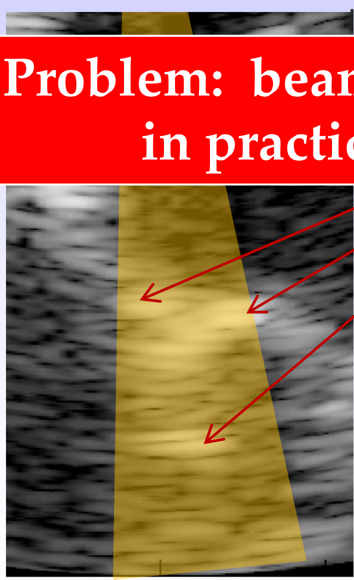
- Recent developments in medical treatment typically imply increasing the number of transducer elements involved in each imaging cycle
- Amount of raw data that needs to be transmitted and processed grows significantly, effecting machinery size and power consumption
- By reducing sampling and processing rate, we may achieve significant reduction of data size - this implies potential reduction of machinery size ar

Our Approach:
Integrate Xampling and beamforming



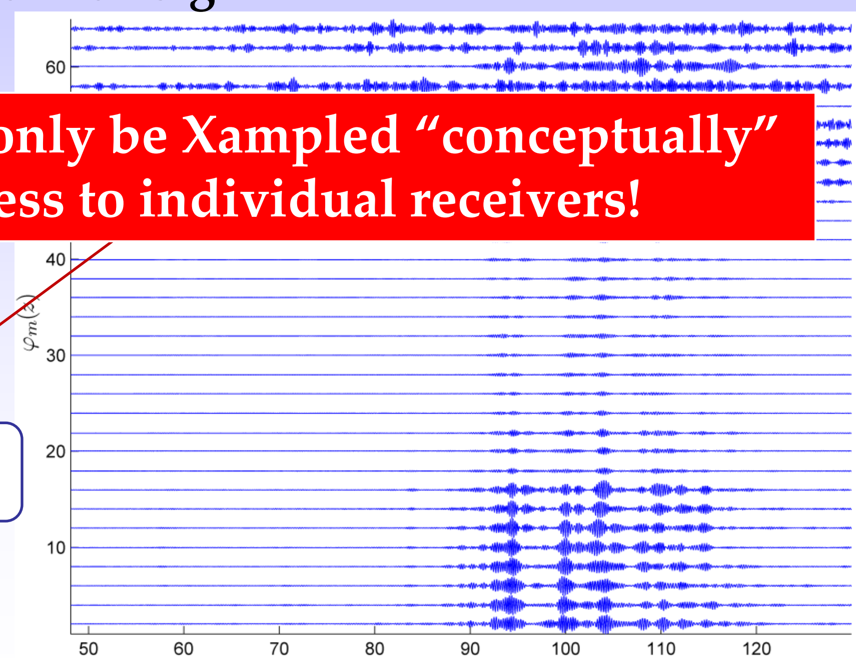
Ultrasound and Xampling

- Possible approach (does not work in practice….): Replace Nyquist rate sampling by Xampling, then reconstruct signals and apply beamforming
- Problems:
  - Low SNR: erroneous parameter extraction by sub-Nyquist scheme
  - Reflections from a relatively wide region: complicated algorithm for matching pulses across signals
- **Proposed solution - Xample the beamformed signal**

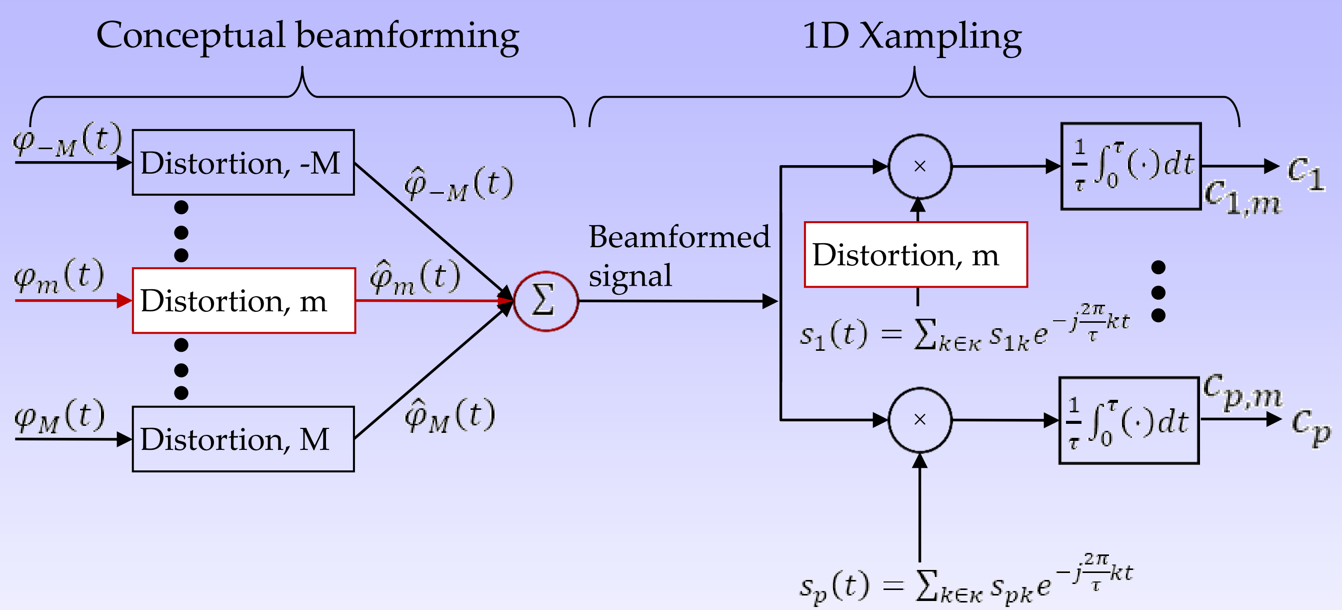


Problem: beamformed signal may only be Xampled "conceptually" in practice – we only have access to individual receivers!

Noise

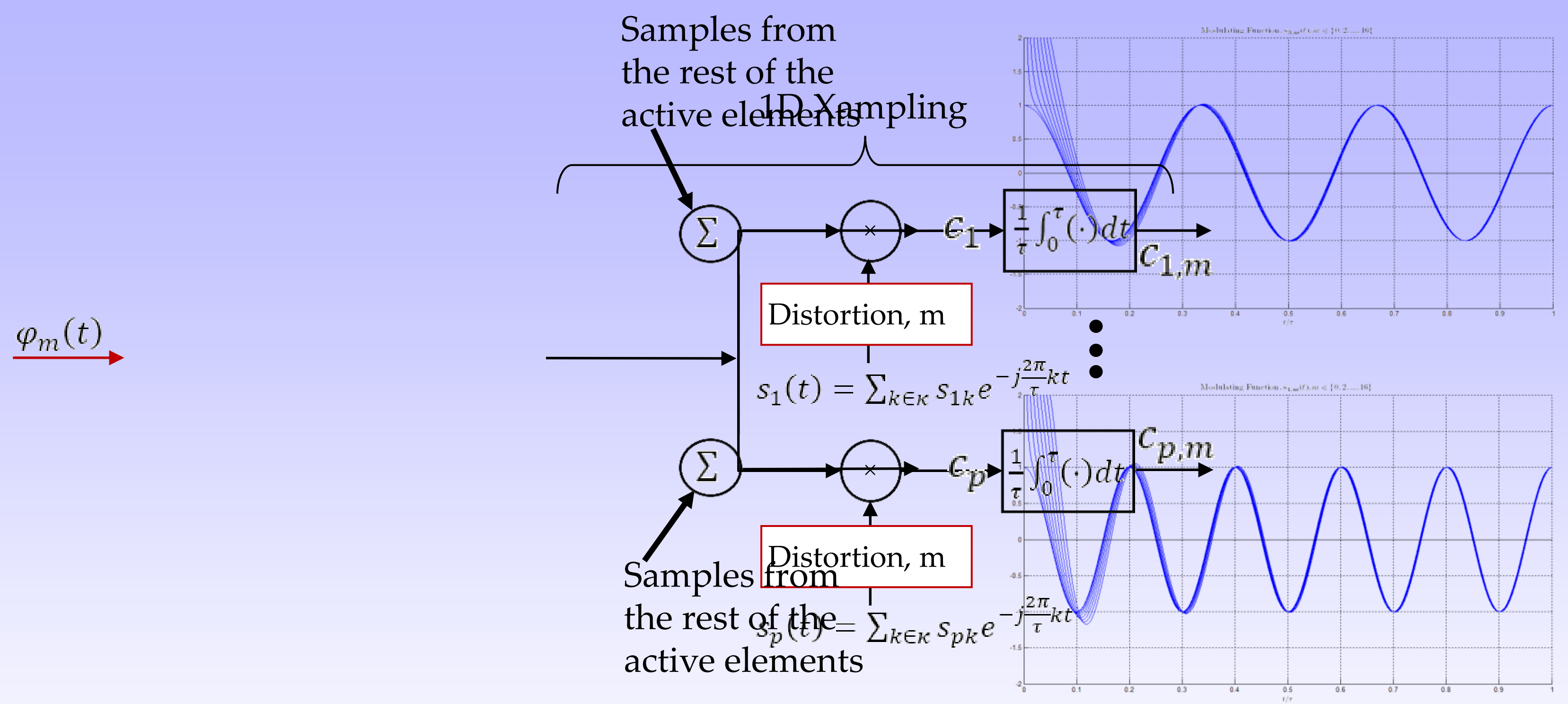


Compressed Beamforming Scheme



- Scheme combines signals from multiple elements for SNR improvement.
- Similar to beamforming techniques used in standard ultrasound imaging.
- Here, the beamforming is moved to the compressed domain – samples at output corresponds to the beamformed signal.

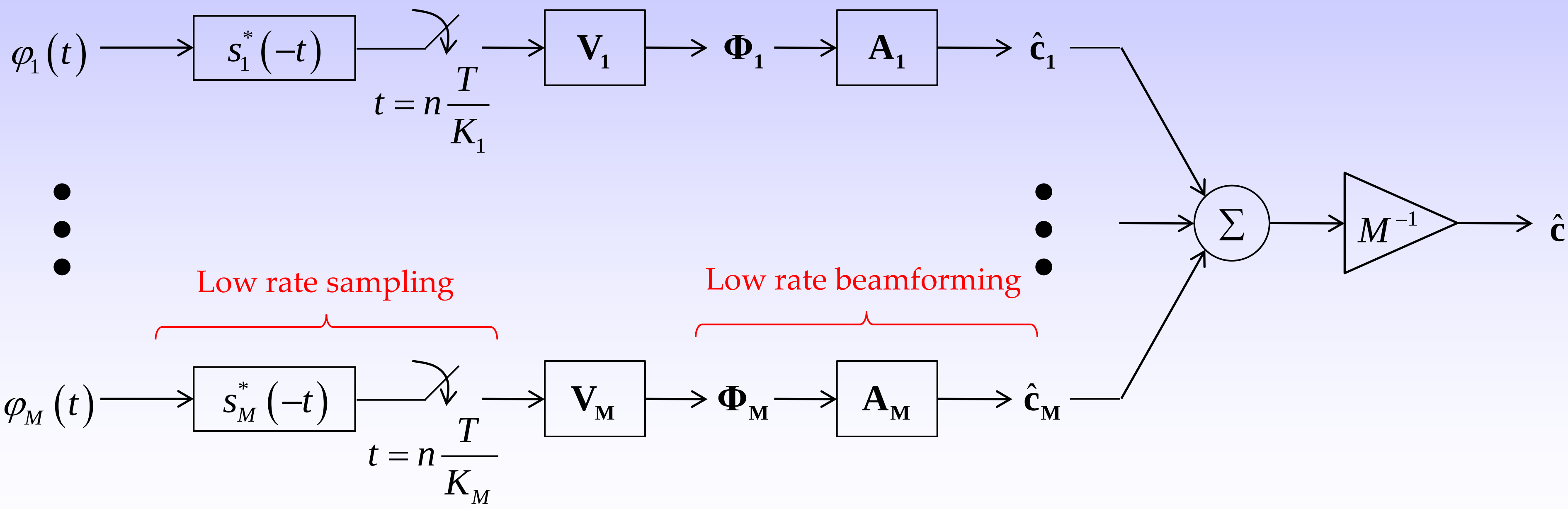
Compressed Beamforming Scheme



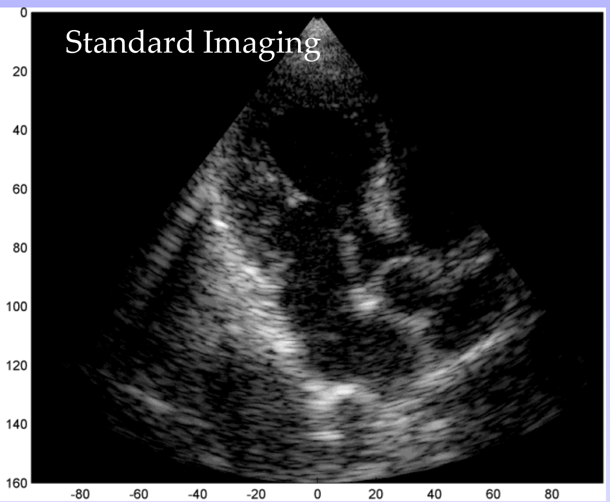
Applying receiver-dependent distortions to two of the modulating kernels

Digital Compressed Beamforming

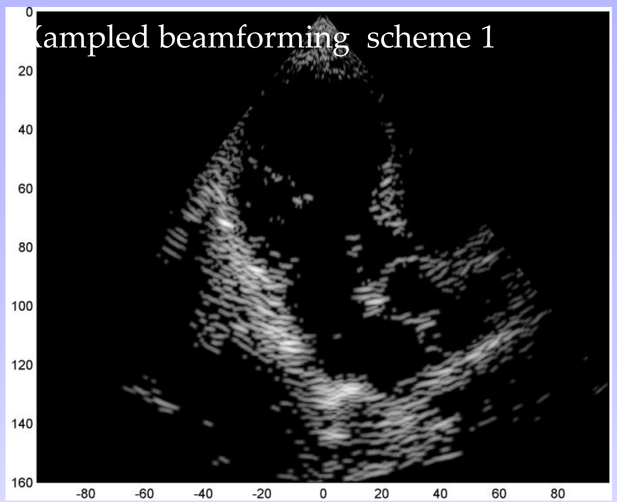
- Using some algebraic manipulations we can show that the same affect can be obtained digitally
- Use existing schemes to extract extended set of Fourier series coefficients (e.g. Sum of Sincs or multichannel bank) and then apply appropriate linear transform on the coefficients



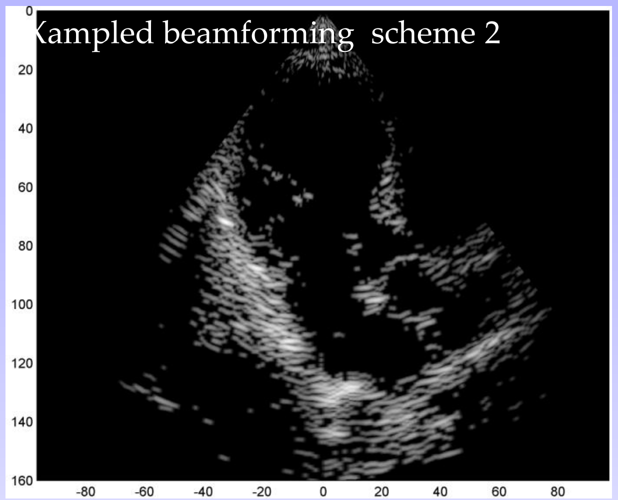
Results



1662 real-valued samples, per sensor per image line



200 real-valued samples, per sensor per image line (assume L=25 reflectors per line)



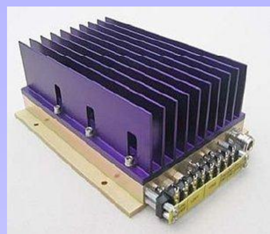
232 real-valued samples, per sensor per image line (average *)

- Xampling results in an error in the peaks with standard deviation being 0.42mm.
- We obtain a more than 7-fold reduction in sample rate.

* Applying 2nd scheme – Max. number of samples (for some line angles & sensor indexes) - 266

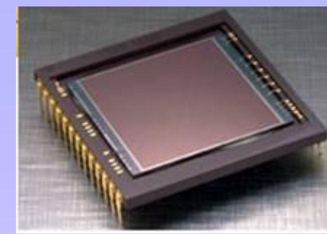
Nonlinear Sampling

Power amplifiers



Optical modulators

CCD arrays

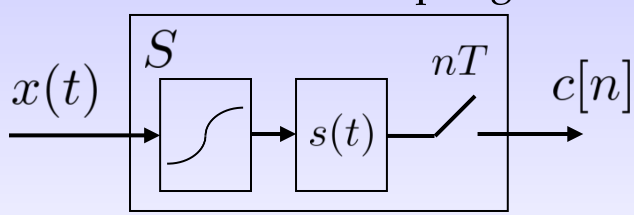


Companding

- Results can be extended to include many classes of nonlinear sampling

Example:

Nonlinear Sampling



Michaeli & Eldar, '12

- In particular we have extended these ideas to phase retrieval problems where we recover signals from samples of the Fourier transform magnitude (Candes et. al., Szameit et. al., Shechtman et. al.)
- Many applications in optics: recovery from partially coherent light, crystallography, subwavelength imaging and more

Quadratic Compressed Sensing

Shechtman, Eldar, Szameit and Segev, '11

$$\min_a \|a\|_0 \quad \text{subject to } |a^* M_u a - y_u| \le \epsilon$$

- Define a matrix $X := aa^*$
- Look for X that is:
  - Rank 1
  - Row sparse
  - Consistent with the measurements
  - PSD

$$\begin{aligned}
&\operatorname*{argmin}_X \; Rank(X) \; s.t. \\
&\qquad \sum_a \left(\sum_b X_{ab}^{\;2} \right)^{1/2} \le \zeta \\
&\qquad \left| tr(M_u X) - y_u \right| \le \varepsilon \;\; \forall u \in U \\
&\qquad X \ge 0
\end{aligned}$$

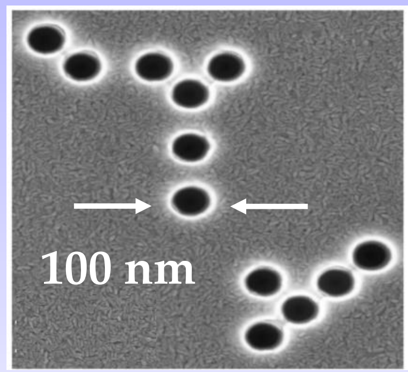
Fazel, Hindi, Boyd 03

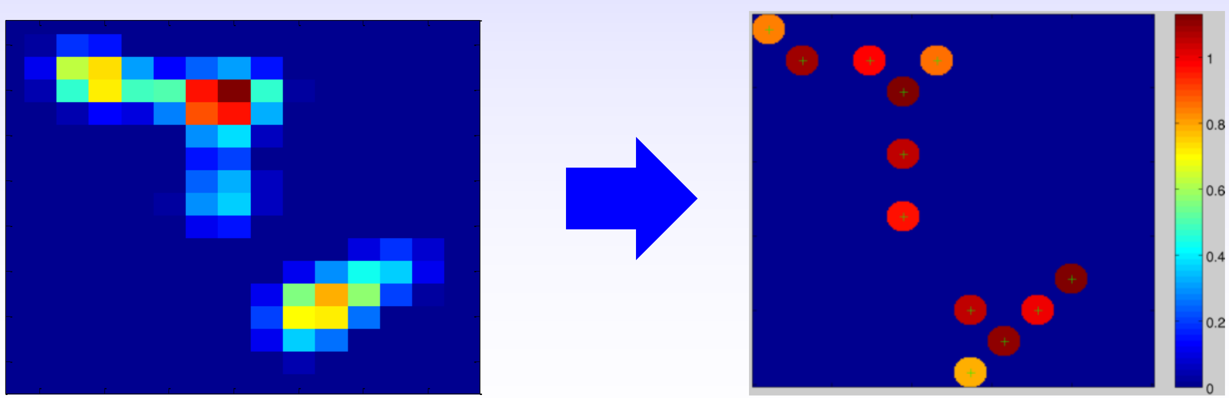
- In practice we replace Rank*(X)* with log det *(X+b I)* and solve iteratively
- Can generalize the approach to more general nonlinearities and use efficient greedy methods *(Beck and Eldar 2012)*

Phase Retrieval

Szameit *et al.*, Nature Photonics, '12

- Subwavelength Coherent Diffractive Imaging:
  Sub-wavelength image recovery from highly truncated Fourier spectrum
- Quadratic CS: based on SDP-relaxation and log-det approximation





Conclusions

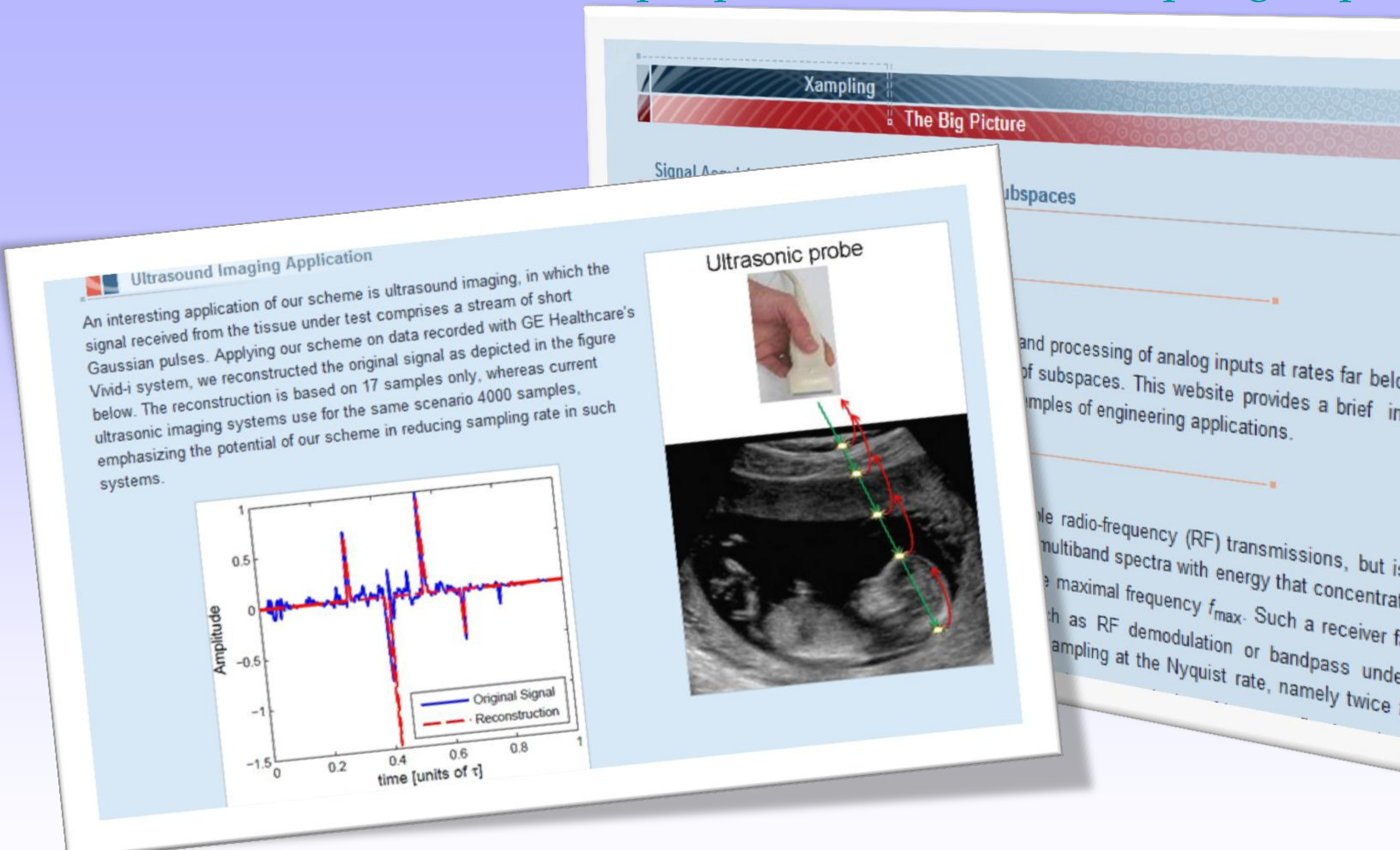
- Compressed sampling and processing of many signals
- Wideband sub-Nyquist samplers in hardware
- Union of subspaces: broad and flexible model
- Practical and efficient hardware
- Many applications and many research opportunities: extensions to other analog and digital problems, robustness, hardware …

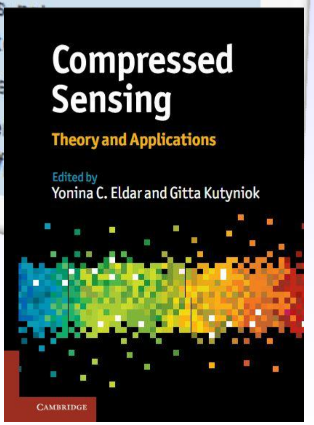
Exploiting structure can lead to a new sampling paradigm which combines analog + digital

More details in:
M. Duarte and Y. C. Eldar, "Structured Compressed Sensing: From Theory to Applications," Review for TSP.
M. Mishali and Y. C. Eldar, "Xampling: Compressed Sensing for Analog Signals", book chapter available at http://webee.technion.ac.il/Sites/People/YoninaEldar/books.html

Xampling Website

webee.technion.ac.il/people/YoninaEldar/xampling_top.html





Y. C. Eldar and G. Kutyniok, "Compressed Sensing: Theory and Applications", Cambridge University Press, 2012

Thank you