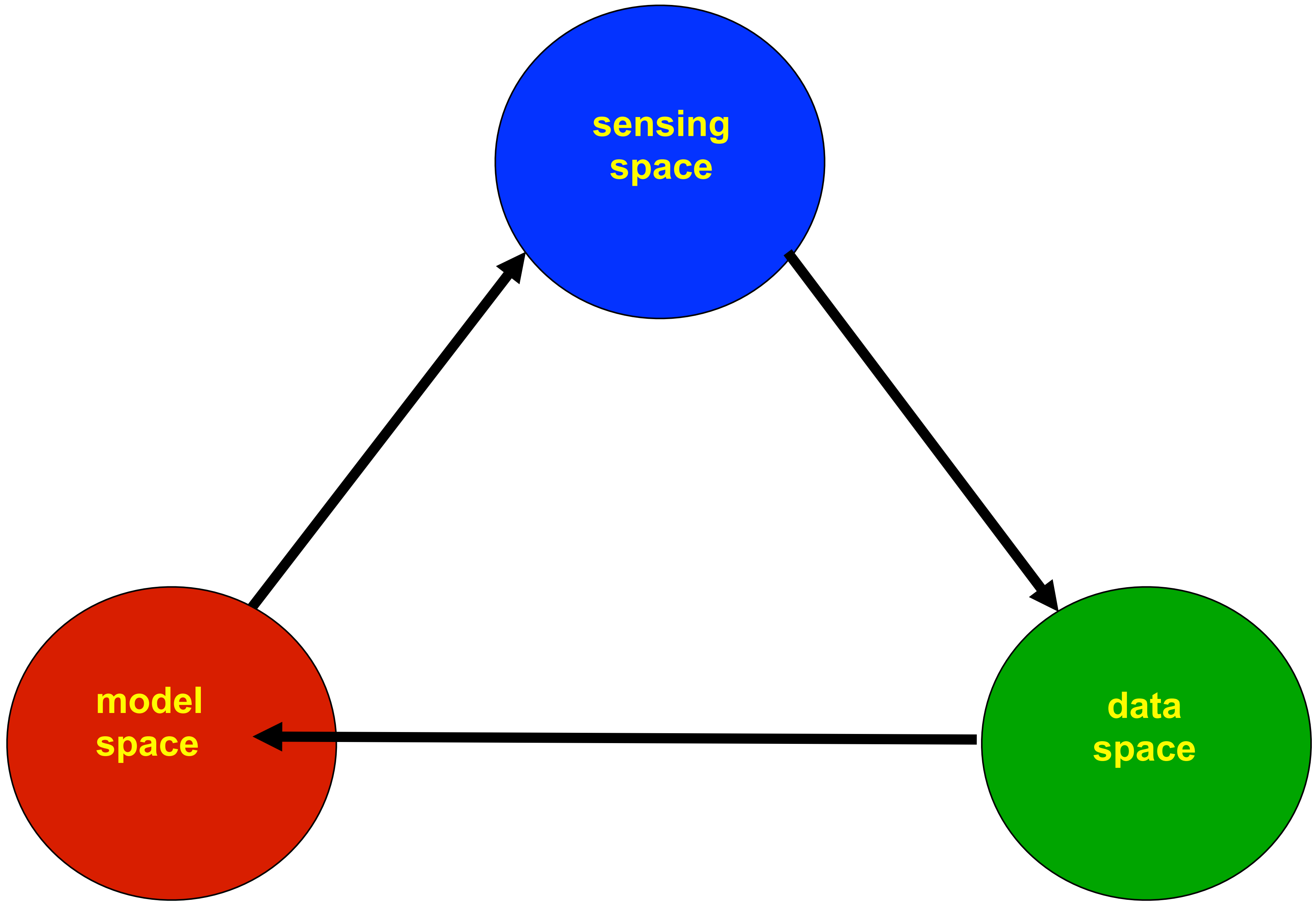
Adaptive Sensing and Active Learning

SSP Workshop August 7, 2012

Rob Nowak www.ece.wisc.edu/~nowak

BIGDATA: An Interactive Approach

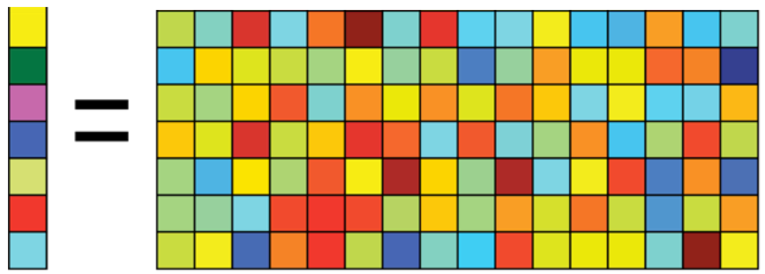
$\mathcal{Y}$: possible measurements/experiments

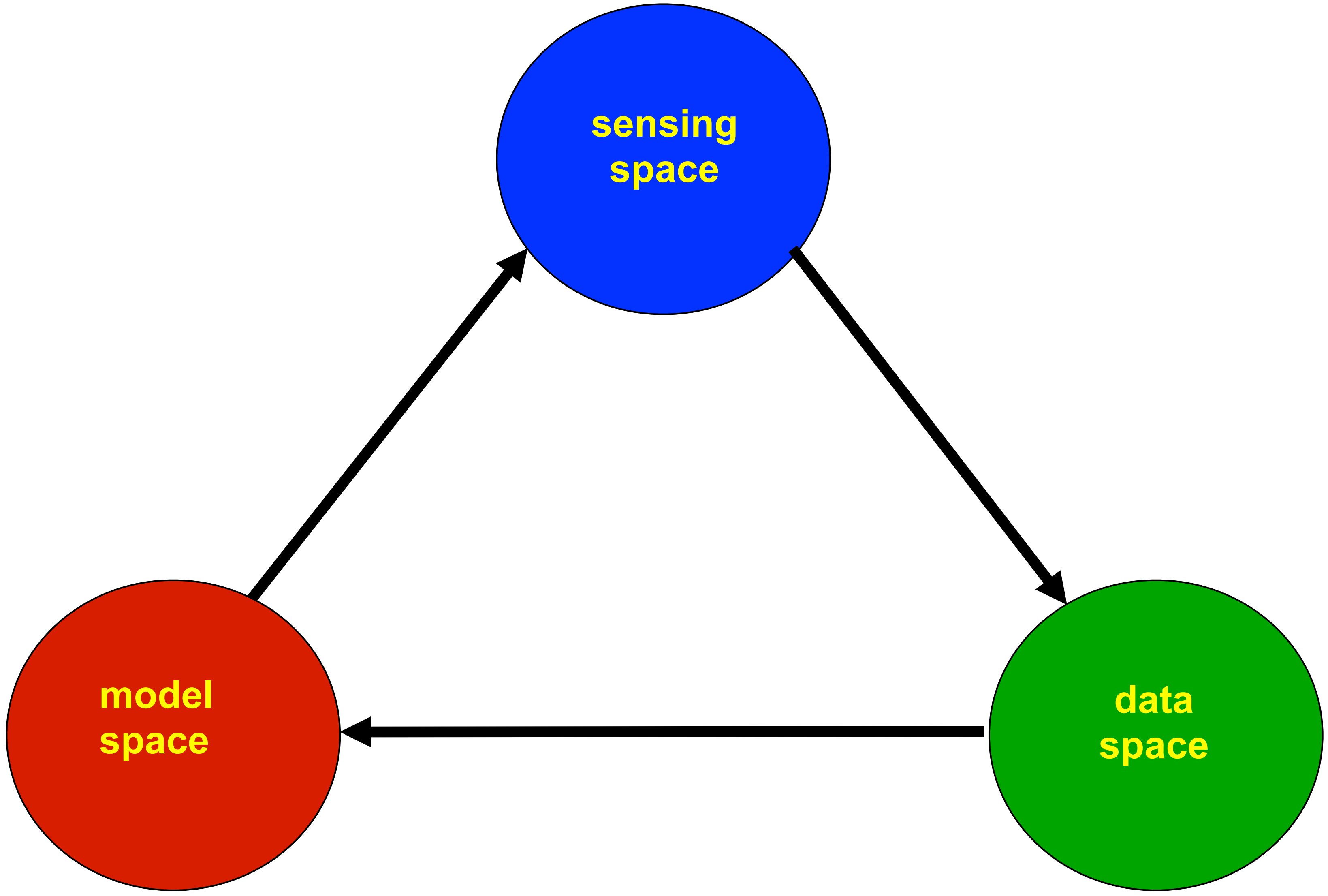


$\mathcal{X}$: models/hypotheses
under consideration

$y_1(x), y_2(x), \ldots$: information/data

Sparse Signals



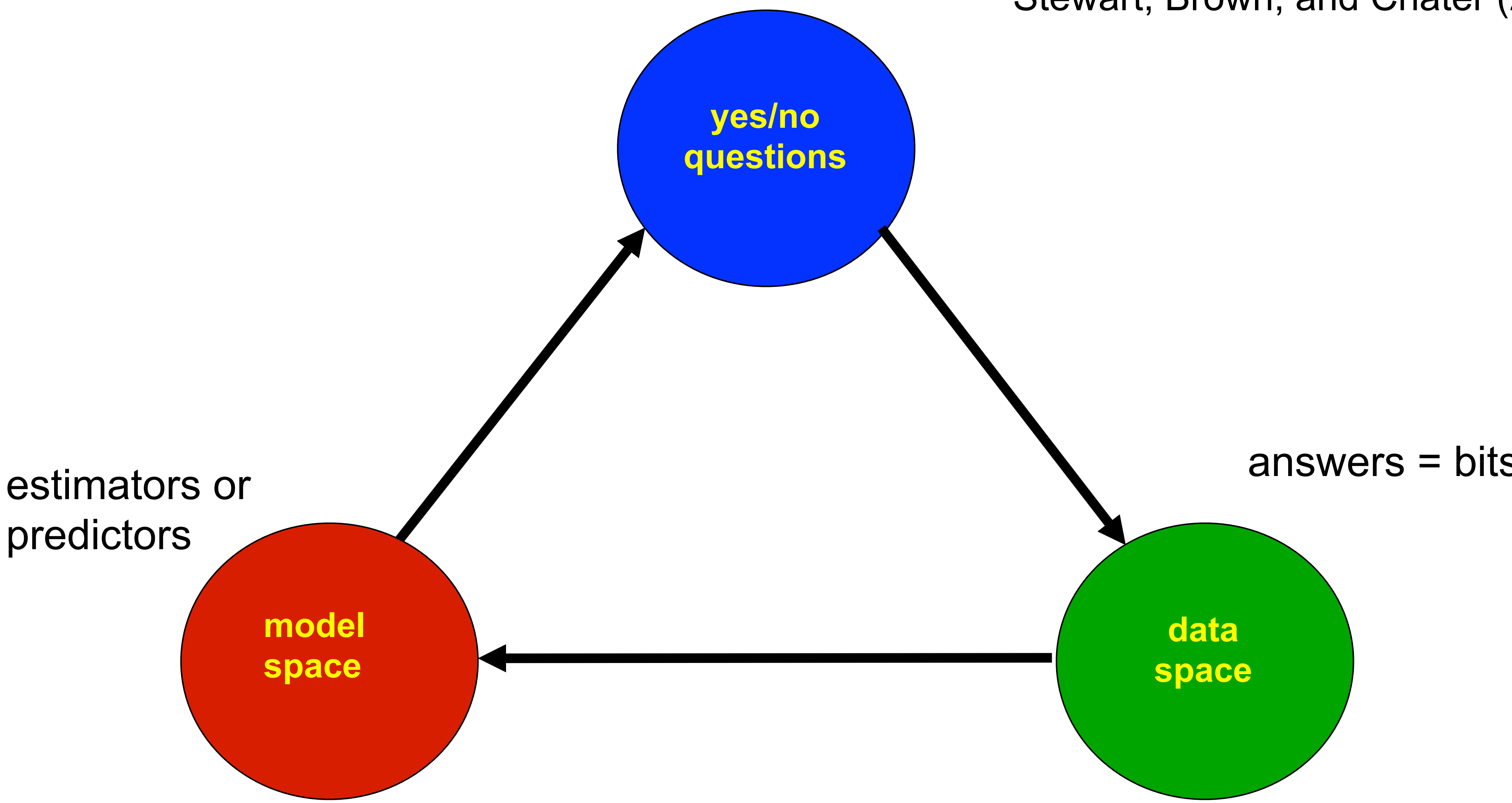


J. Haupt, R. Castro and RN (2011), J. Haupt, R. Castro, R. Baraniuk, and RN (2012)

Humans as Sensors

Humans are much more reliable and consistent at making comparative judgements, than in giving numerical ratings or evaluations

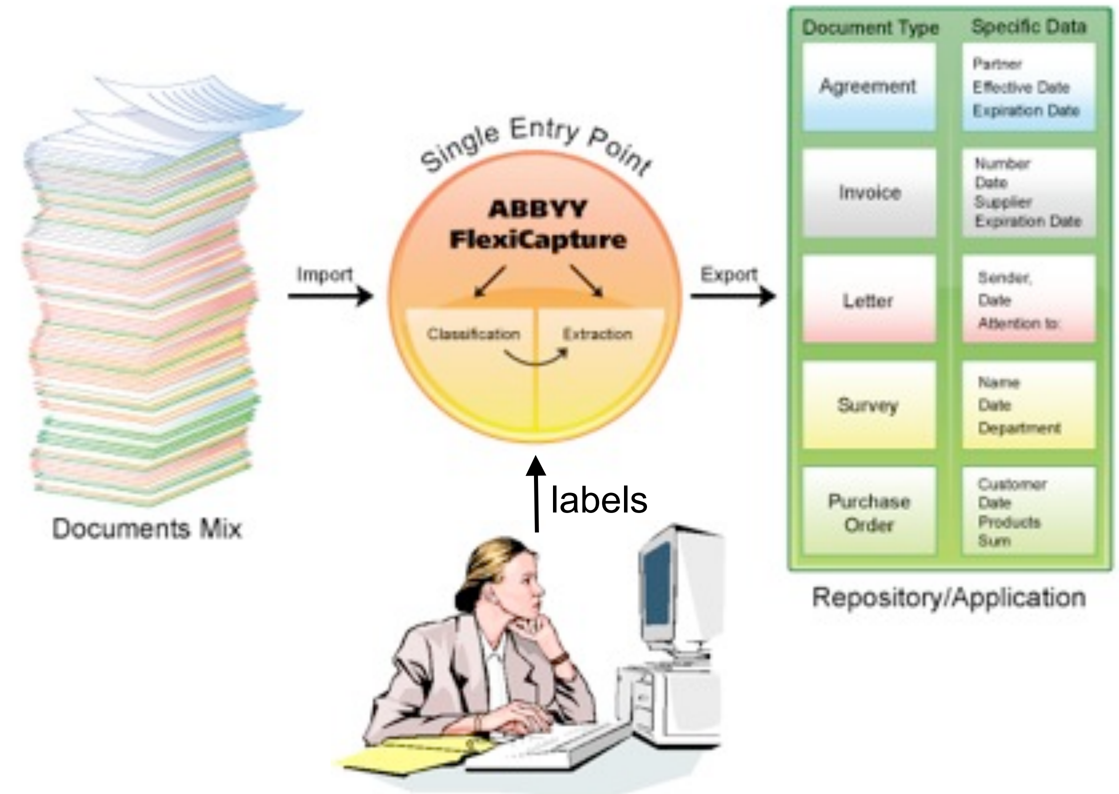
Bijmolt and Wedel (1995)
Stewart, Brown, and Chater (2005)



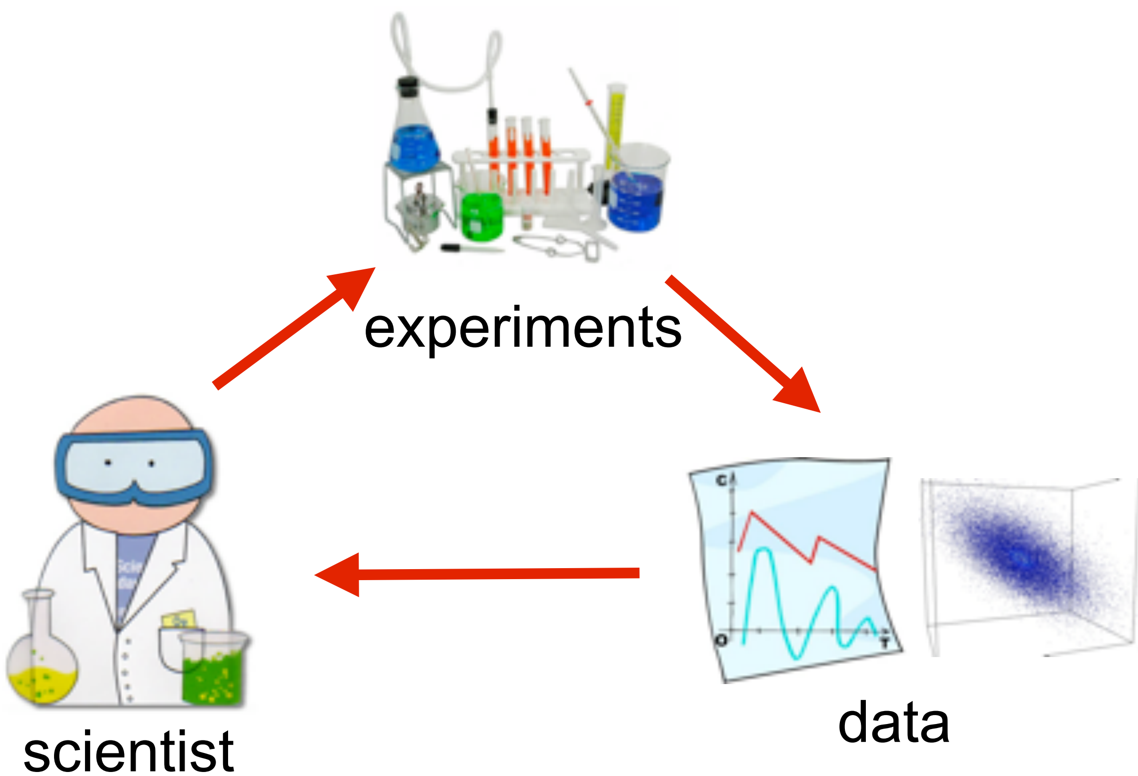
Machine Learning from Human Judgements

Recommendation Systems

Document Classification



Optimizing Experimentation



Challenge:

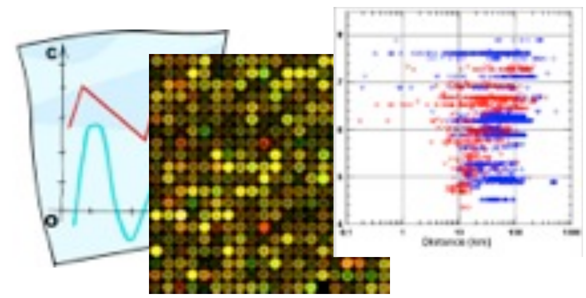
Computing is cheap, but human assistance/guidance is expensive

Goal:

Optimize such systems with as little human involvement as possible

Machine Learning (Passive)

Raw unlabeled data



$X_1, X_2, X_3, \ldots$

$(X_1, Y_1), (X_2, Y_2), (X_3, Y_3), \ldots$

Labeled data

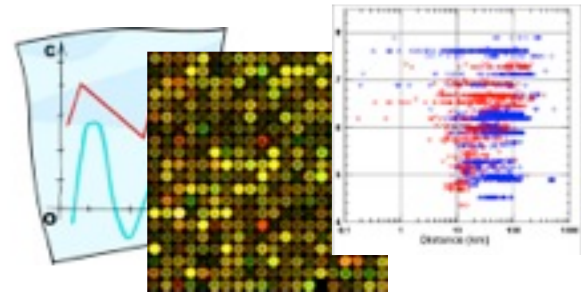
passive learner

expert/oracle
provides labels

rule for predicting Y from X

Active Learning

Raw unlabeled data



$X_1, X_2, X_3, \ldots$

machine requests labels
for **selected** data

$(X_1, ?)$

(X_1, Y_1)

X_2 identical or very similar to $X_1 \ldots$
no need to ask for label

$(X_3, ?)$

(X_3, Y_3)

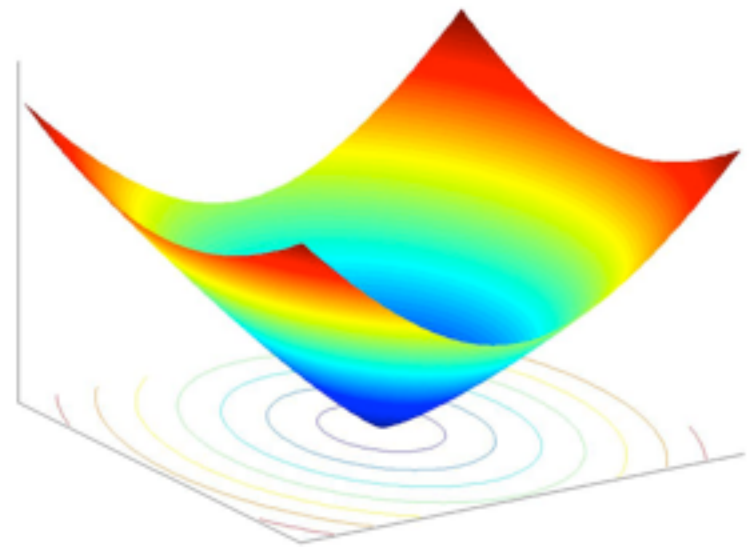
active learner

expert/oracle
provides labels

rule for predicting Y from X

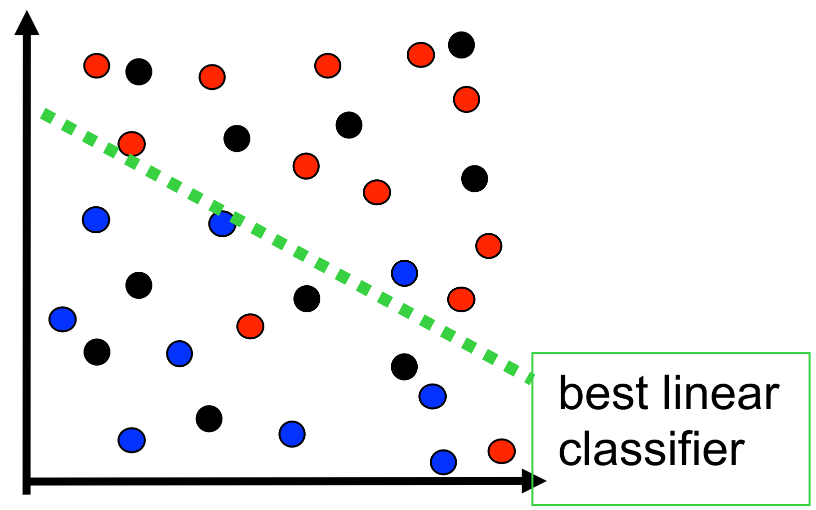
Outline

1. Derivative Free Optimization using Human Subjects

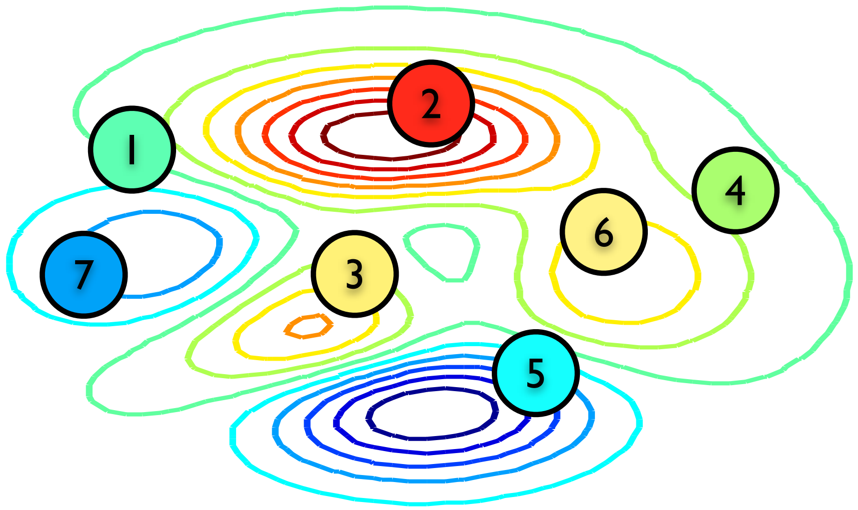


minimizing a convex function

2. Binary Classification via Active Learning

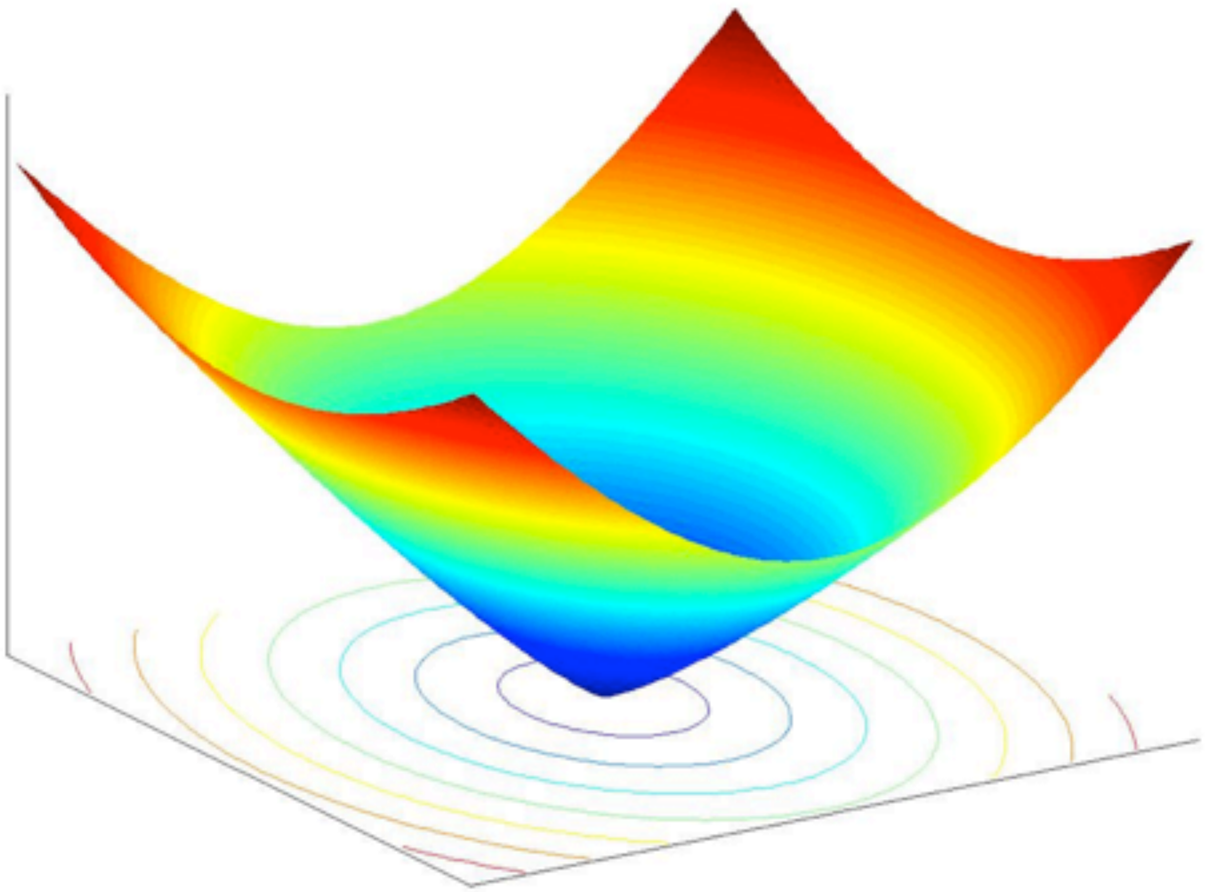


3. Ranking from Pairwise Comparisons



ranking or embedding objects in a low-dimensional space

Optimization Based on Human Judgements

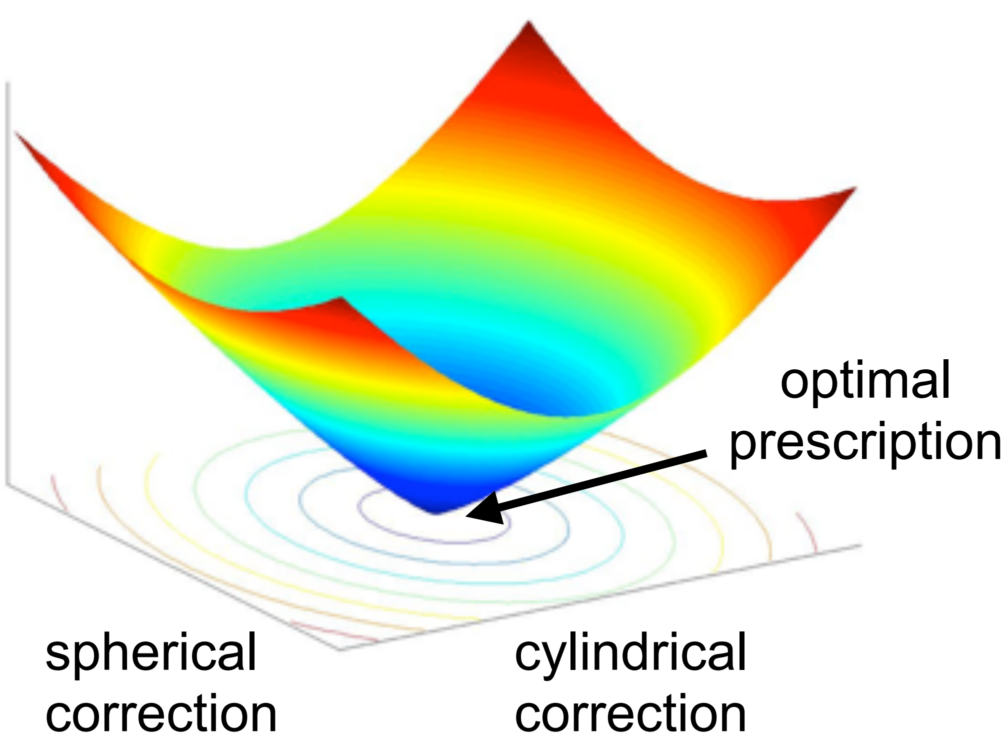


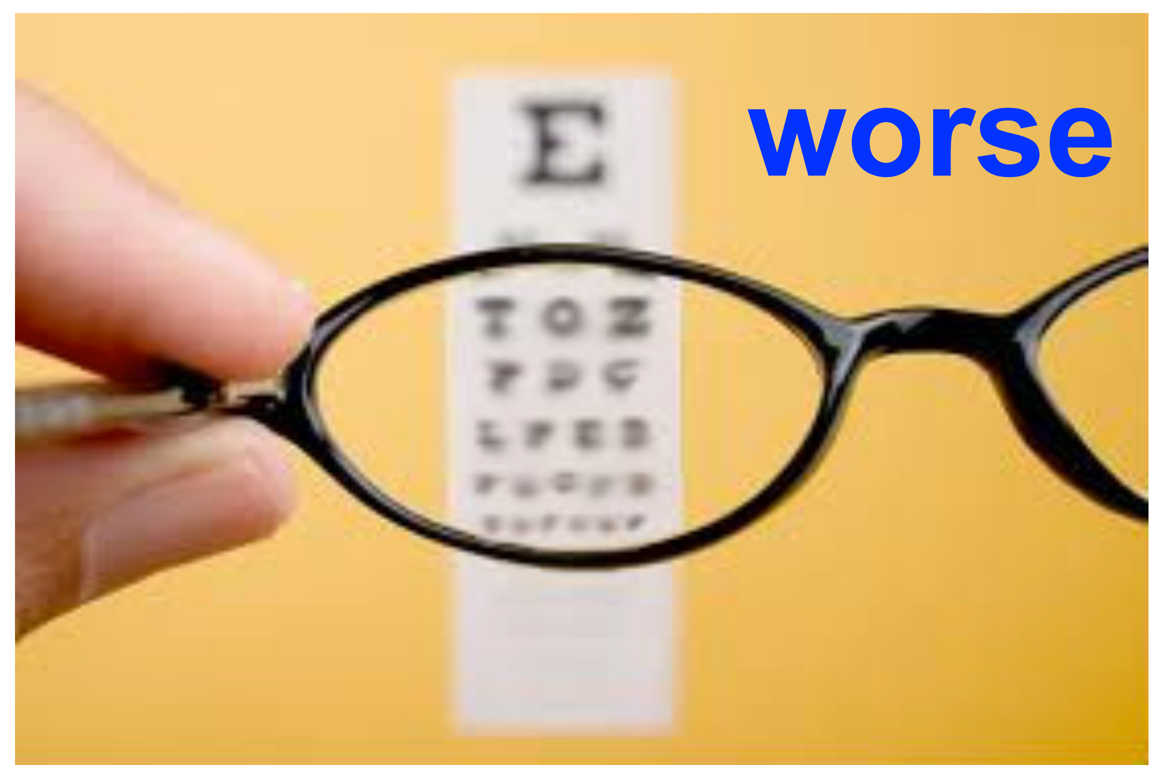
convex function to be minimized

Human oracles can provide function values or comparisons, but not function gradients

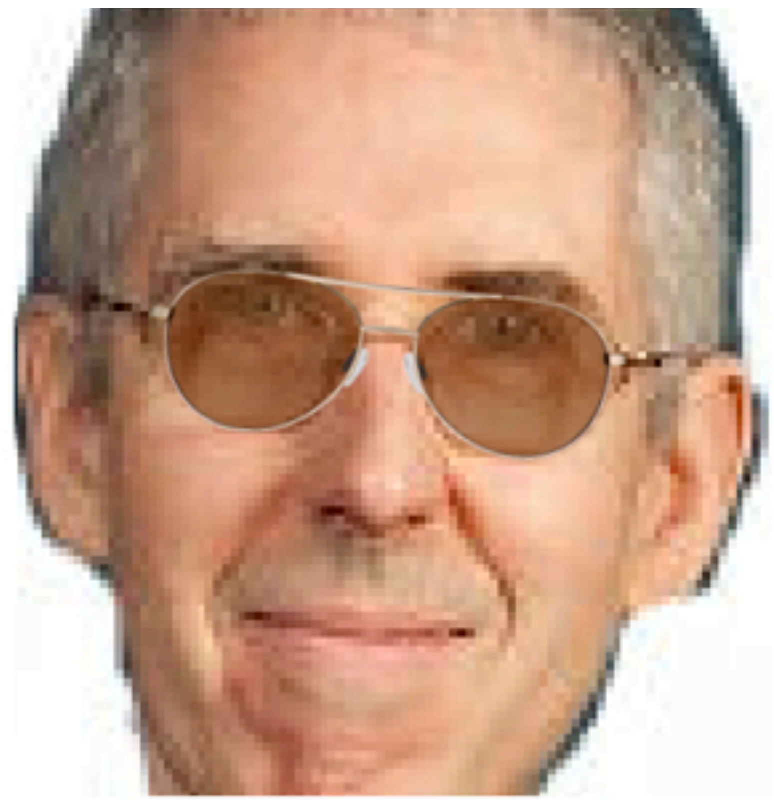
Methods that don't use gradients are called Derivative Free Optimization (DFO)

A Familiar Application





In the Future... Custom Frame Optimization

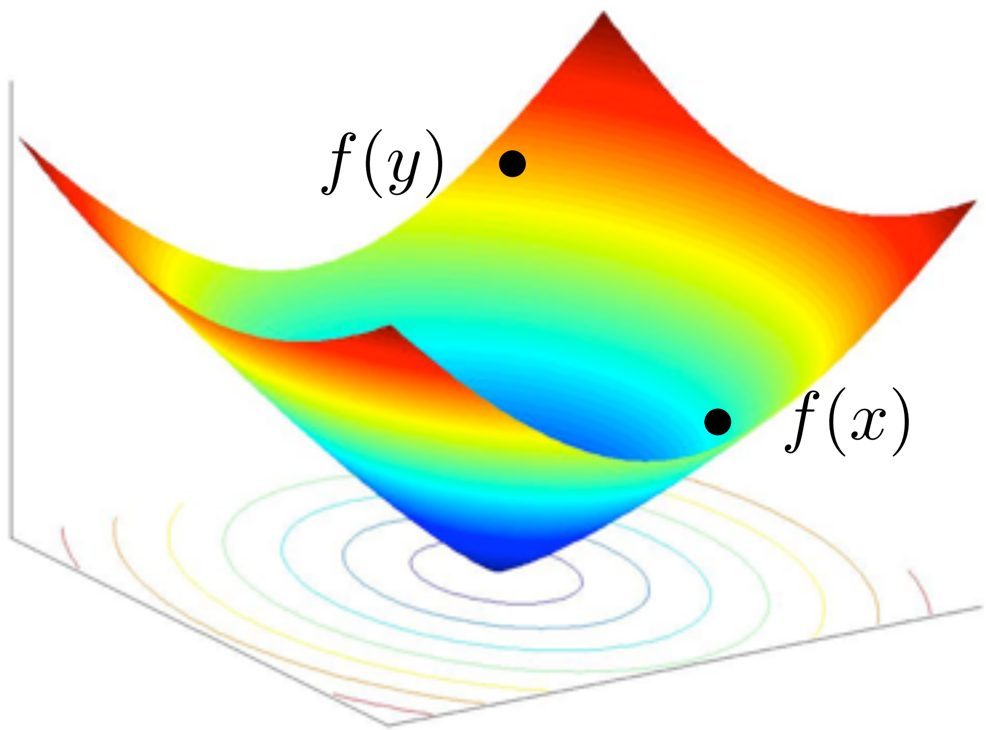


better **worse**

optimization dimensions: frame size, material, shape, color, lens tint

Optimization based on Pairwise Comparisons

Assume that the (unknown) function f to be optimized is strongly convex with Lipschitz gradients.



The function will be minimized by asking pairwise comparisons of the form:

$$\text{Is } f(x) \; > \; f(y) \quad ?$$

Assume that the answers are probably correct: for some $\delta > 0$

$$\mathbb{P}\left(\text{answer} = \text{sign}(f(x) - f(y))\right) \; \geq \; \frac{1}{2} \; + \; \delta$$

Optimization based on Pairwise Comparisons

Optimization with Pairwise Comparisons

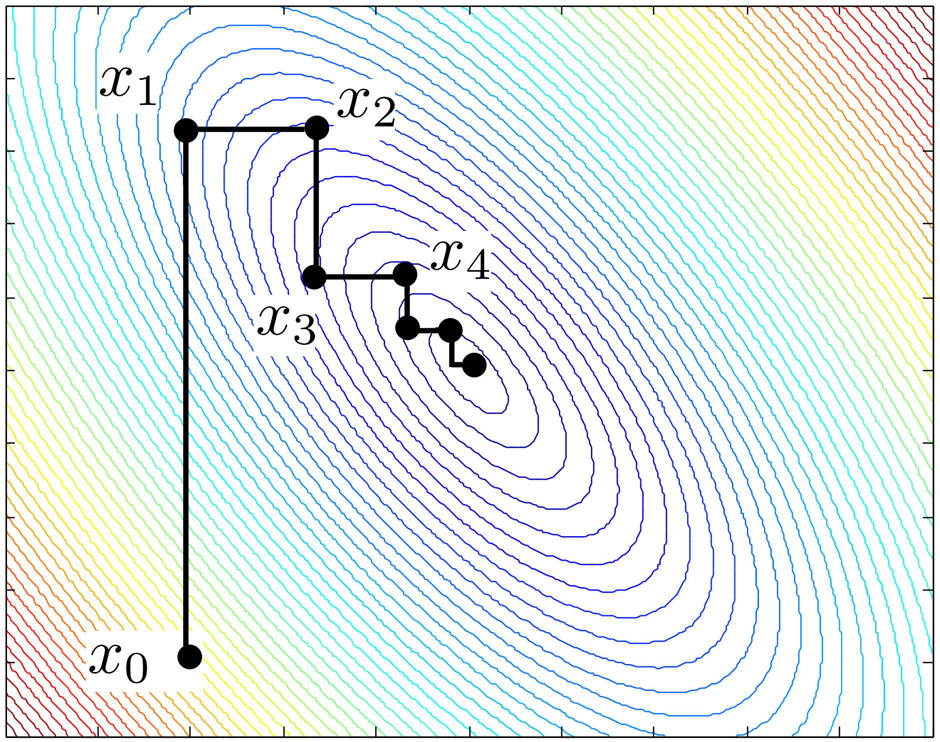
initialize: x_0 = random point

for $n = 0, 1, 2, \ldots$

1) select one of d coordinates uniformly at random and consider line along coordinate that passes x_n

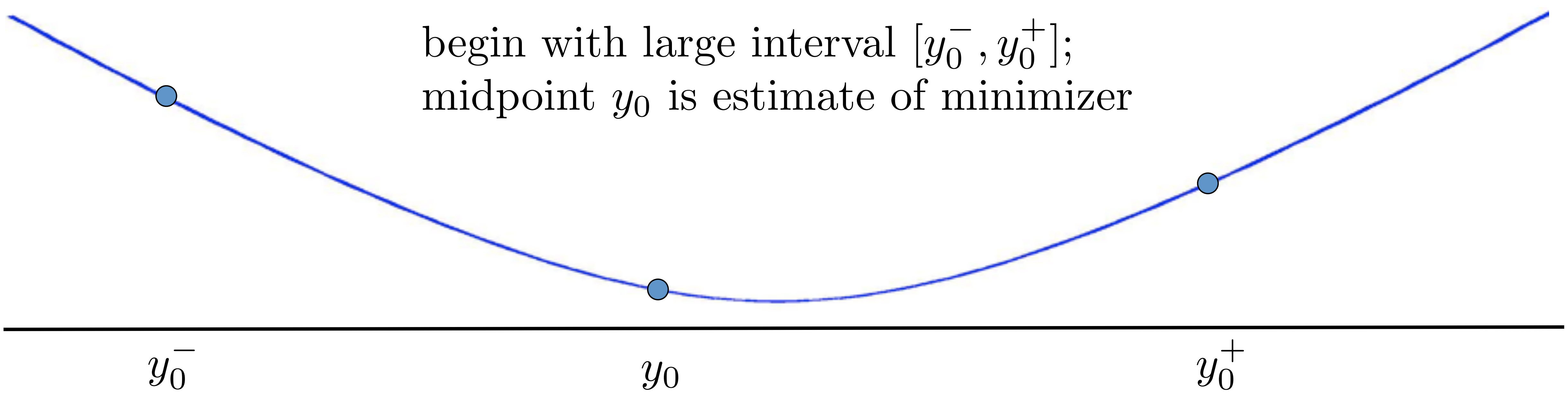
2) minimize along coordinate using pairwise comparisons and binary search

3) x_{n+1} = approximate minimizer



line search iteratively reduces interval containing minimum

begin with large interval $[y_0^-, y_0^+]$;
midpoint y_0 is estimate of minimizer



Optimization based on Pairwise Comparisons

Optimization with Pairwise Comparisons

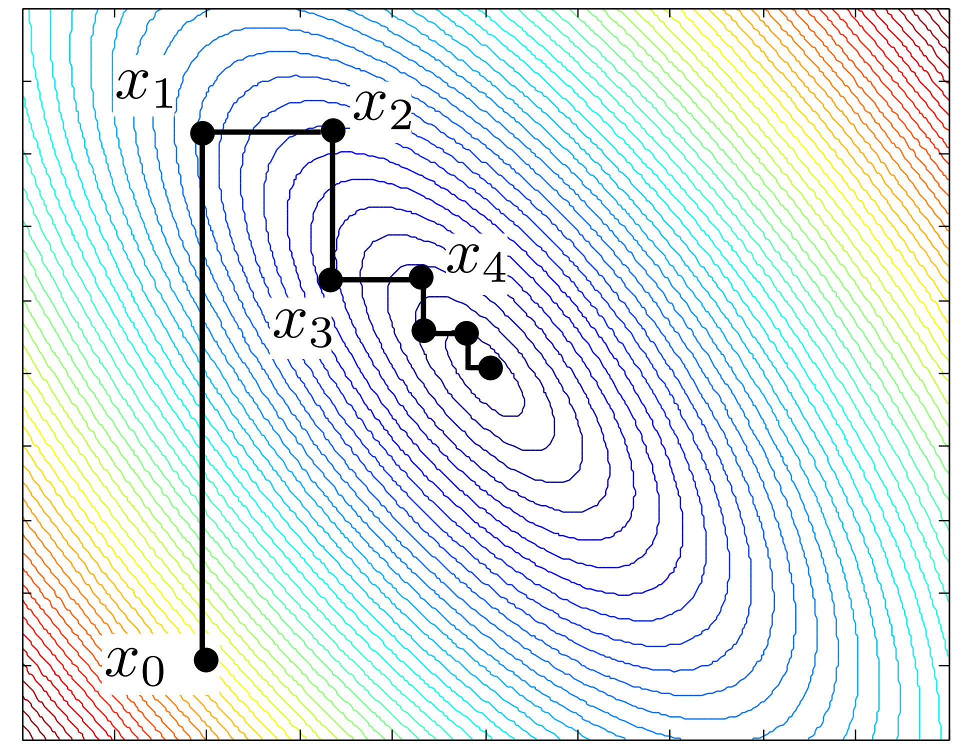
initialize: x_0 = random point

for $n = 0, 1, 2, \ldots$

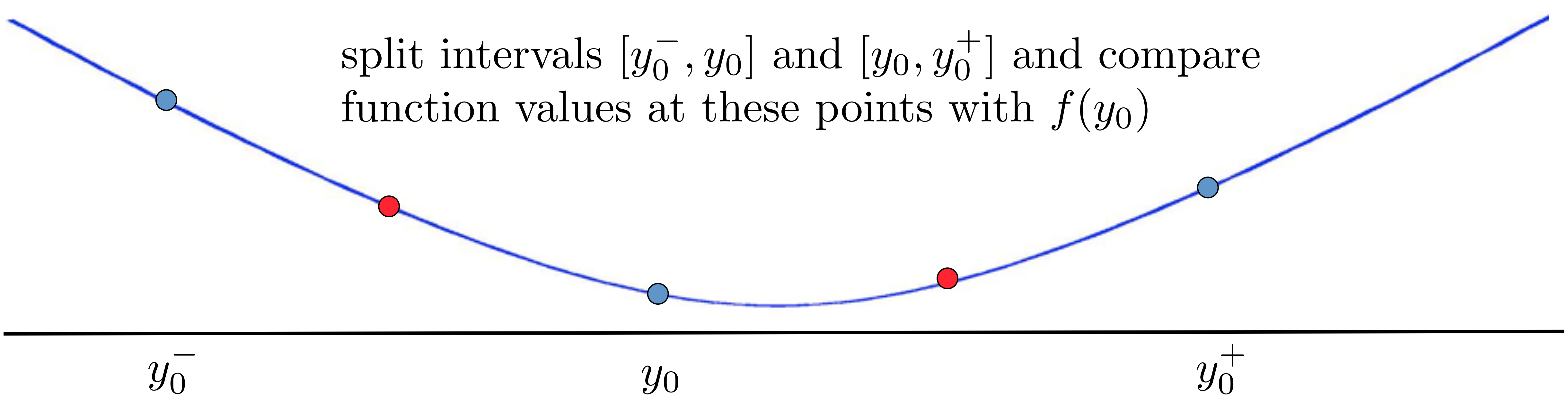
1) select one of d coordinates uniformly at random and consider line along coordinate that passes x_n

2) minimize along coordinate using pairwise comparisons and binary search

3) x_{n+1} = approximate minimizer



line search iteratively reduces interval containing minimum



Optimization based on Pairwise Comparisons

Optimization with Pairwise Comparisons

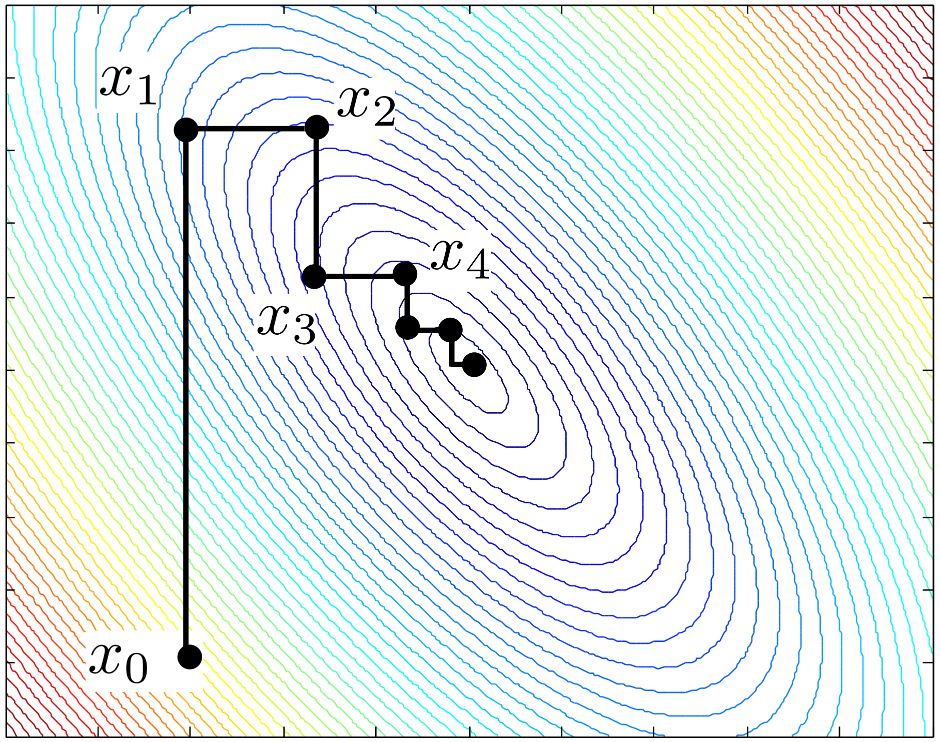
initialize: x_0 = random point

for $n = 0, 1, 2, \ldots$

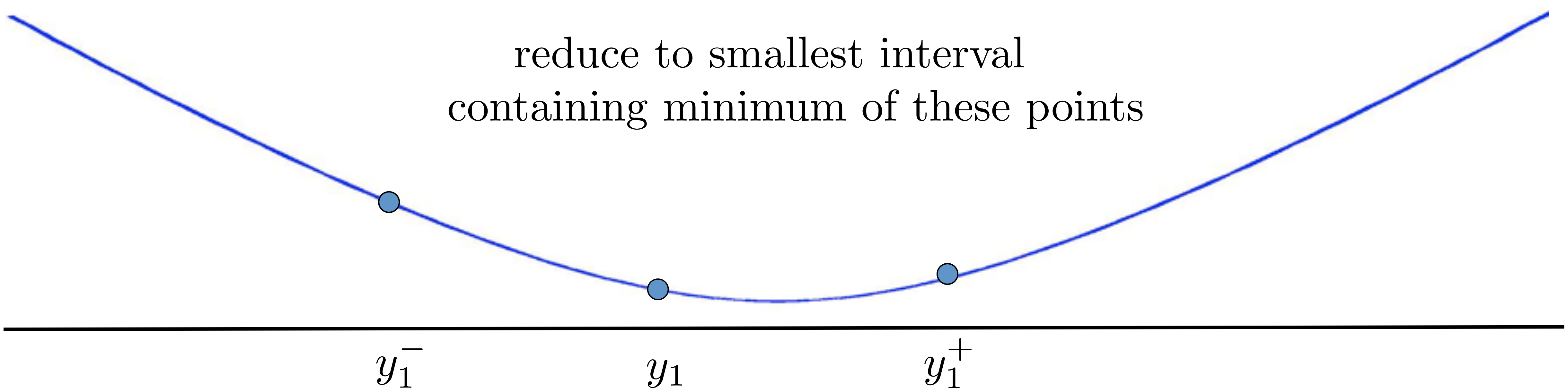
1) select one of d coordinates uniformly at random and consider line along coordinate that passes x_n

2) minimize along coordinate using pairwise comparisons and binary search

3) x_{n+1} = approximate minimizer



line search iteratively reduces interval containing minimum



Optimization based on Pairwise Comparisons

Optimization with Pairwise Comparisons

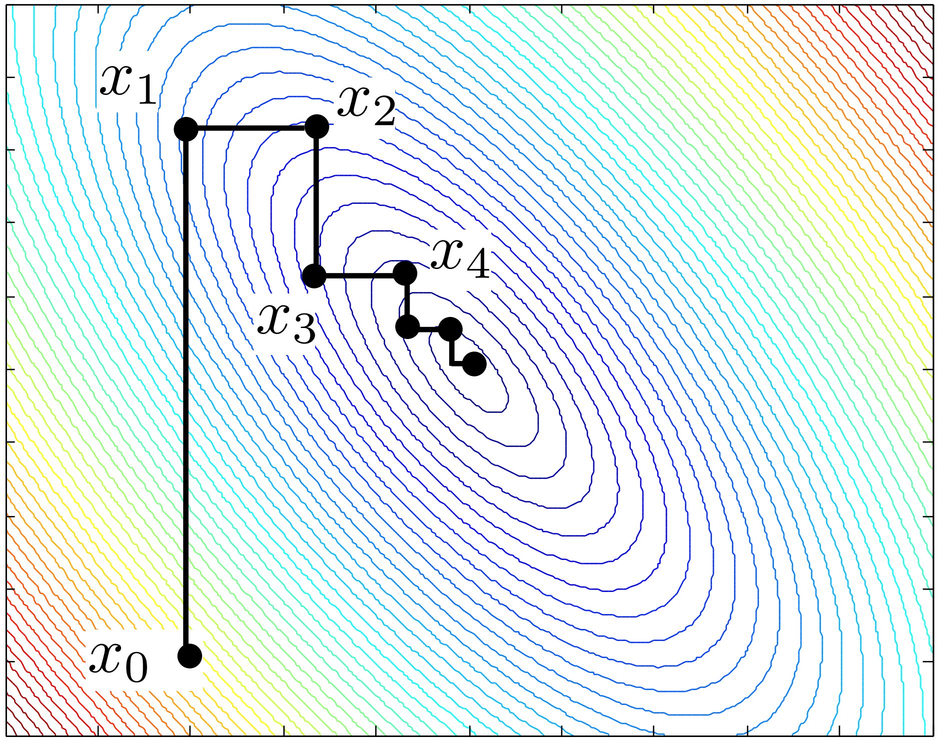
initialize: x_0 = random point

for $n = 0, 1, 2, \ldots$

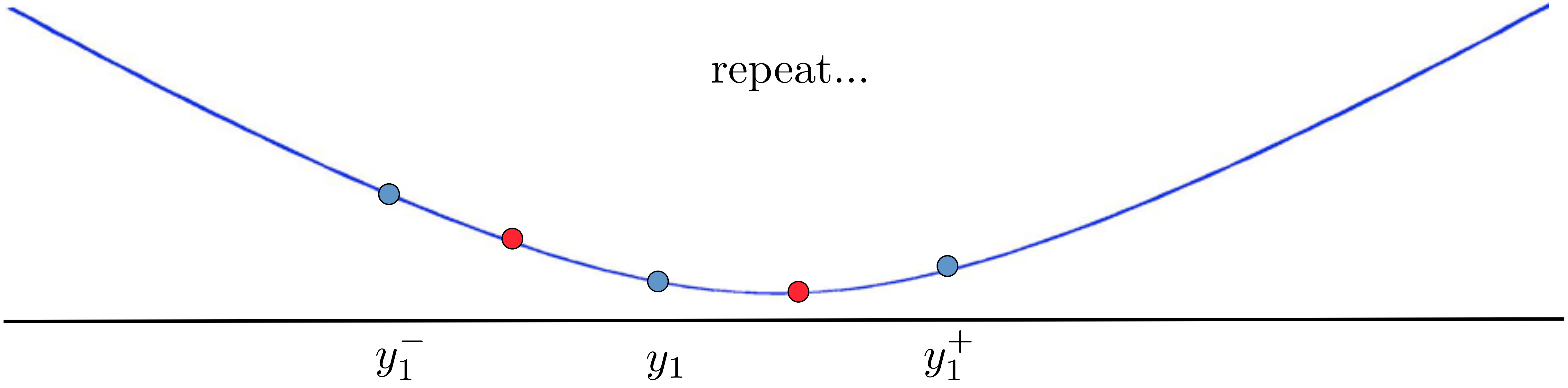
1) select one of d coordinates uniformly at random and consider line along coordinate that passes x_n

2) minimize along coordinate using pairwise comparisons and binary search

3) x_{n+1} = approximate minimizer



line search iteratively reduces interval containing minimum



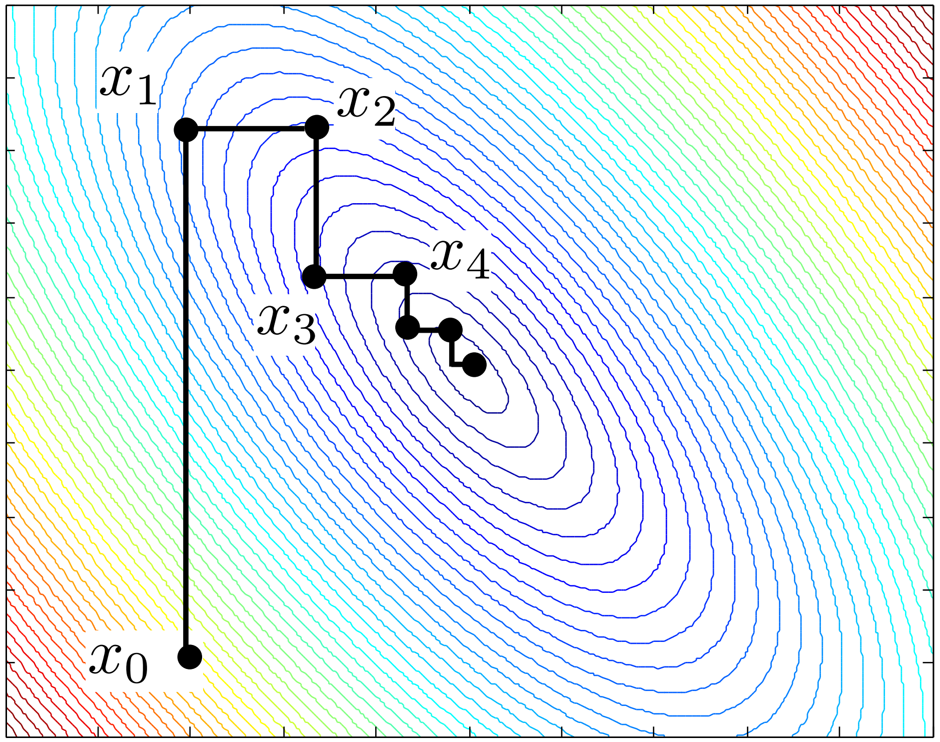
Optimization based on Pairwise Comparisons

Optimization with Pairwise Comparisons

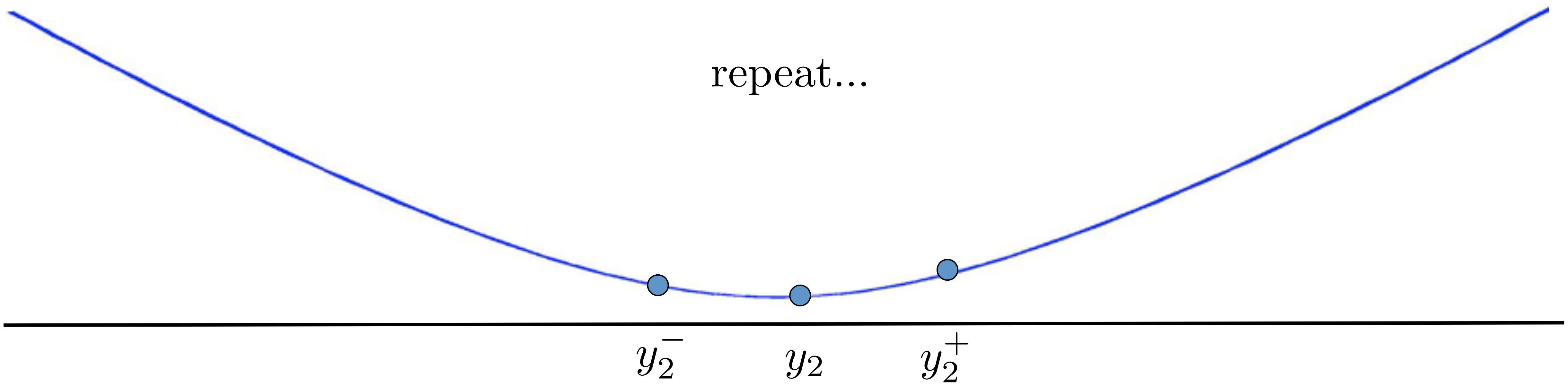
initialize: x_0 = random point

for $n = 0, 1, 2, \ldots$

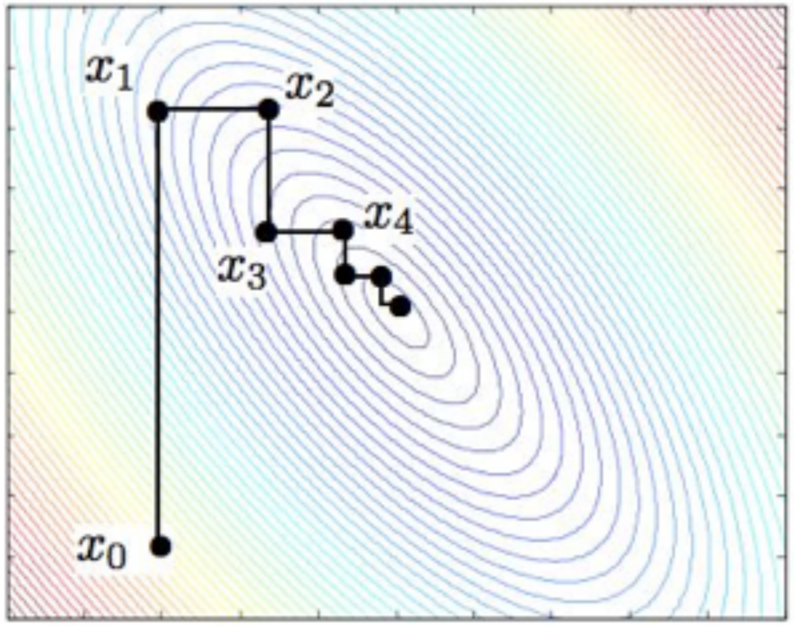
1) select one of d coordinates uniformly at random and consider line along coordinate that passes x_n
2) minimize along coordinate using pairwise comparisons and binary search
3) x_{n+1} = approximate minimizer



line search iteratively reduces interval containing minimum



Convergence Analysis



If we want error $:= \mathbb{E}[f(x_k) - f(x^*)] \leq \epsilon$, we must solve $k \approx d \log \frac{1}{\epsilon}$ line searches (standard coordinate descent bound) and each must be at least $\sqrt{\frac{\epsilon}{d}}$ accurate

Noiseless Case:

each line search requires $\frac{1}{2}\log(\frac{d}{\epsilon})$ comparisons

$\Rightarrow$ total of $n \approx d \log \frac{1}{\epsilon} \log \frac{d}{\epsilon}$ comparisons

$\Rightarrow \epsilon \approx \exp\left(-\sqrt{\frac{n}{d}}\right)$

Noisy Case: probably correct answers to comparisons:

$$\mathbb{P}\left(\text{answer} = \text{sign}(f(x) - f(y))\right) \geq \frac{1}{2} + \delta$$

take majority vote of repeated comparisons to mitigate noise

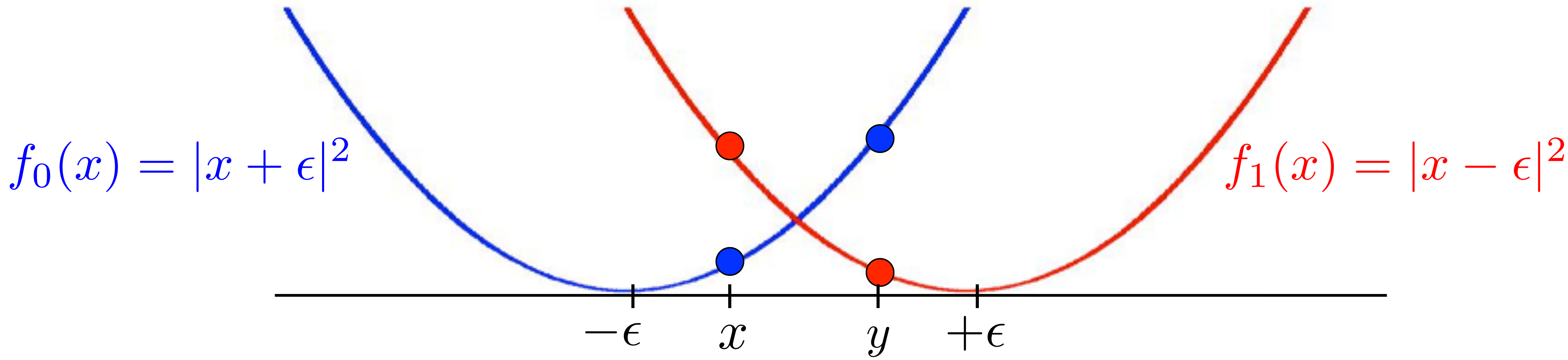
Bounded Noise ($\delta \geq \delta_0 > 0$):

line searches require $C \log \frac{d}{\epsilon}$ comparisons, where $C > 1/2$ depends on $\delta_0 \Rightarrow \epsilon \approx \exp\left(-\sqrt{\frac{n}{d\,C}}\right)$

Unbounded Noise ($\delta \propto |f(x) - f(y)|$):

line searches require $\left(\frac{d}{\epsilon}\right)^2$ comparisons $\Rightarrow \epsilon \approx \sqrt{\frac{d^3}{n}}$

Lower Bounds



For unbounded noise, $\delta \propto |f(x) - f(y)|$, Kullback-Leibler Divergence between response to $f_0(x) > f_0(y)?$ vs. $f_1(x) > f_1(y)?$ is $O(\epsilon^4)$, and KL Divergence between n responses is $O(n\epsilon^4)$

with $\epsilon \sim n^{-1/4}$

- KL Divergence = constant
- squared distance between minima $\sim n^{-1/2}$

$$\Rightarrow \mathbb{P}\left(f(x_n) - f(x^*) \geq n^{-1/2}\right) \geq \text{ constant}$$

matches $O(n^{-1/2})$ upper bound of algorithm

Jamieson, Recht, RN (2012)

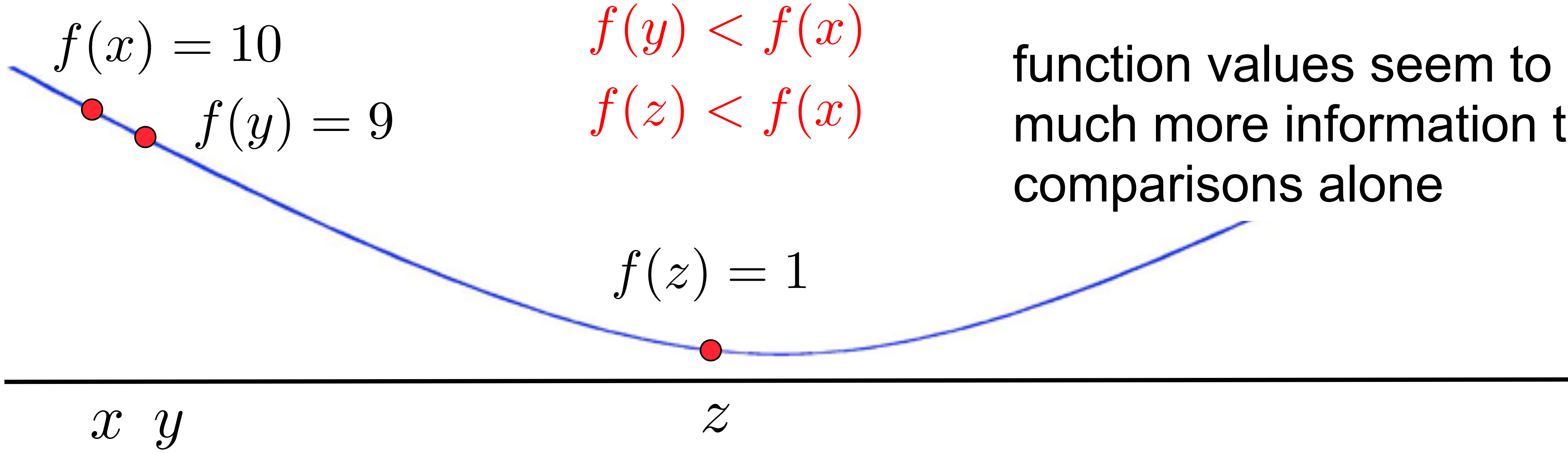
A Surprise

Could we do better with function evaluations (e.g., ratings instead of comparisons)?

suppose we can obtain noisy function evaluations of the form: $f(x) + \text{noise}$

$f(y) < f(x)$

$f(z) < f(x)$



function values seem to provide much more information than comparisons alone

lower bound on optimization error with noisy function evaluations $\sqrt{\frac{d}{n}}$

evaluations give at best a small improvement over comparisons

Jamieson, Recht, RN (2012)

upper bound on optimization error with noisy pairwise comparisons $\sqrt{\frac{d^3}{n}}$

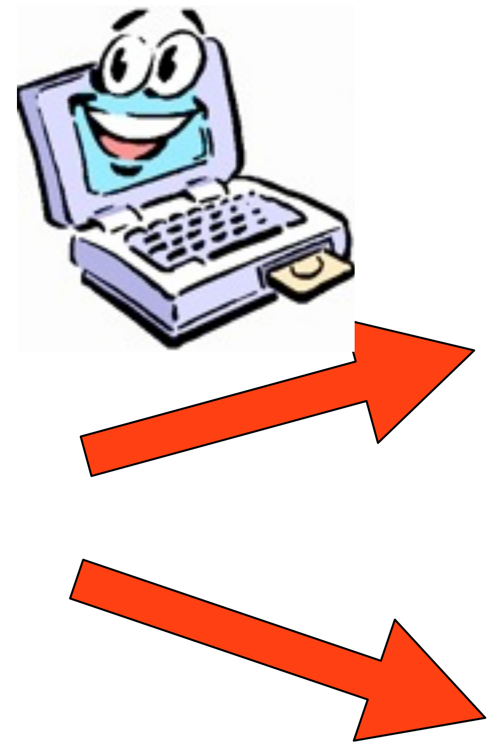
see Agrawal, Dekel, Xiao (2010)
for similar upper bounds for function evals

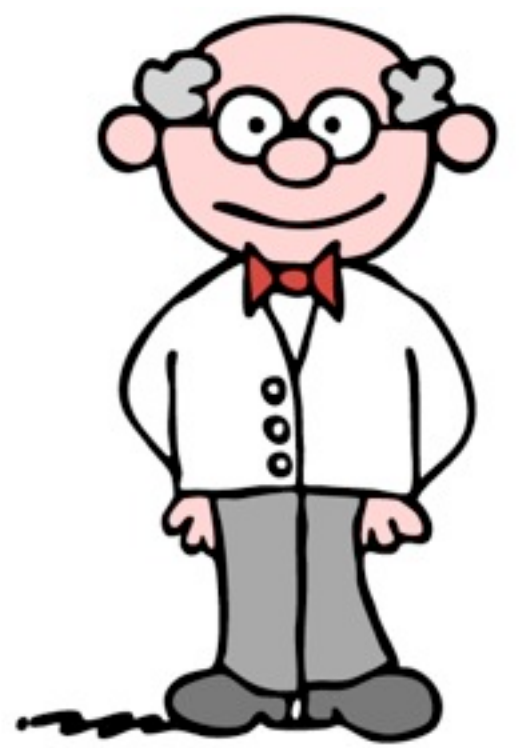
if we could measure noisy gradients (and function is strongly convex), then $O(\frac{d}{n})$ convergence rate is possible

Nemirovski et al 2009

Binary Classification

unlabeled documents



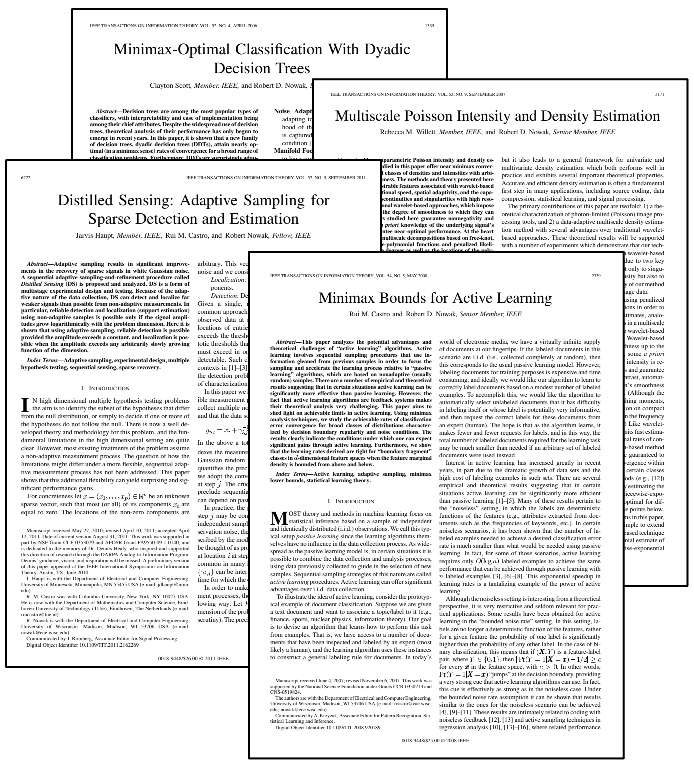


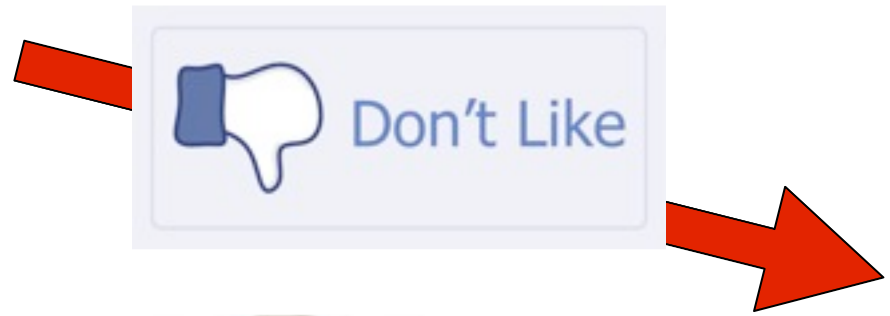
expert/oracle

provides labels to machine learner

Tong and Koller (2001)

A Possible Application





submitted manuscripts

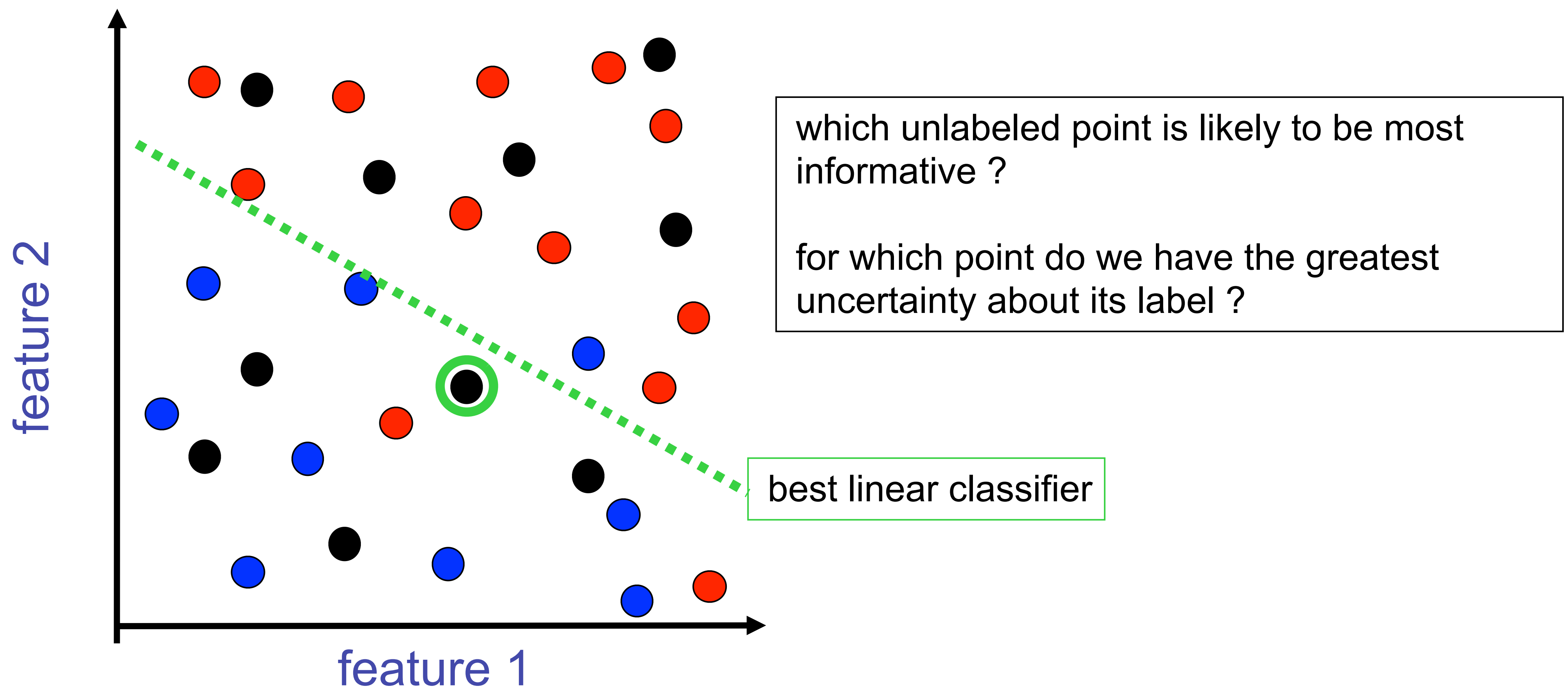
to improve notoriously long review process, we will use Dave's expertise to train a computer to automatically accept/reject submissions

features used by computer

= { # of equations, length of proofs, # of mentions of Shannon, etc }

Active Learning

Learning Problem: Consider a binary prediction problem involving a collection of "classifiers." Each classifier maps points in the "feature-space" (e.g., $\mathbb{R}^d$) to binary labels. The features and labels are governed by an *unknown* distribution P. The goal is to select the classifier that minimizes the probability of misclassification using as few training examples as possible.

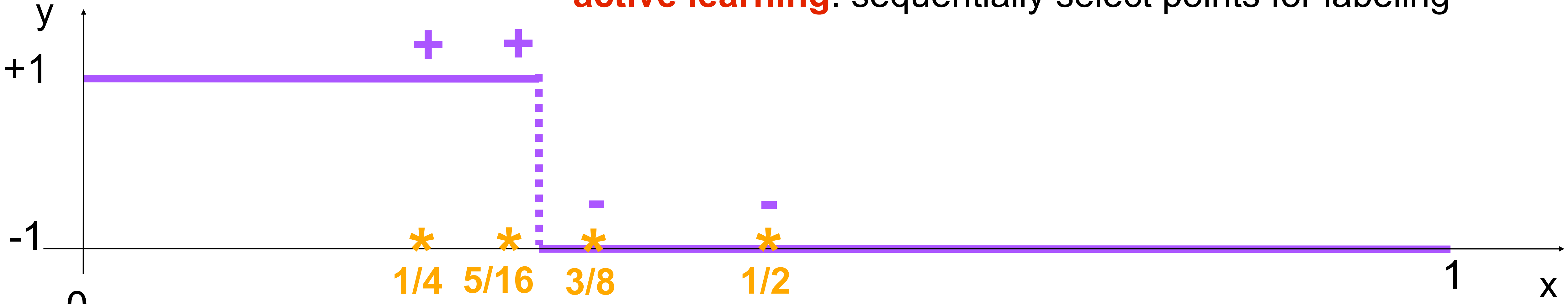


Standard approaches assume training data are obtained prior to learning.

However, some examples are more informative than others, so sequential selection of data can dramatically accelerate learning.

1D Classification - Classic Binary Search

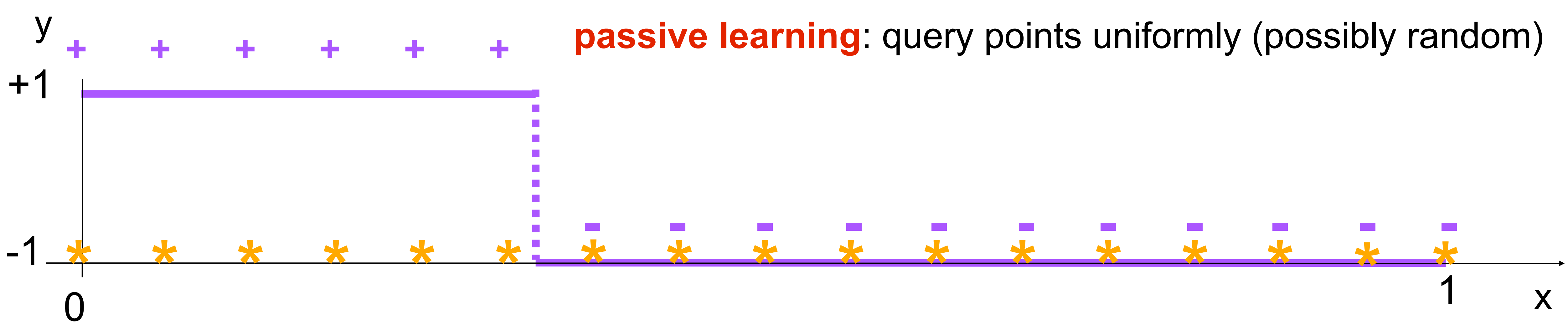
active learning: sequentially select points for labeling



1/3 = 0101…

ϵ-accuracy requires $\log_2 \frac{1}{\epsilon}$ queries

passive learning: query points uniformly (possibly random)

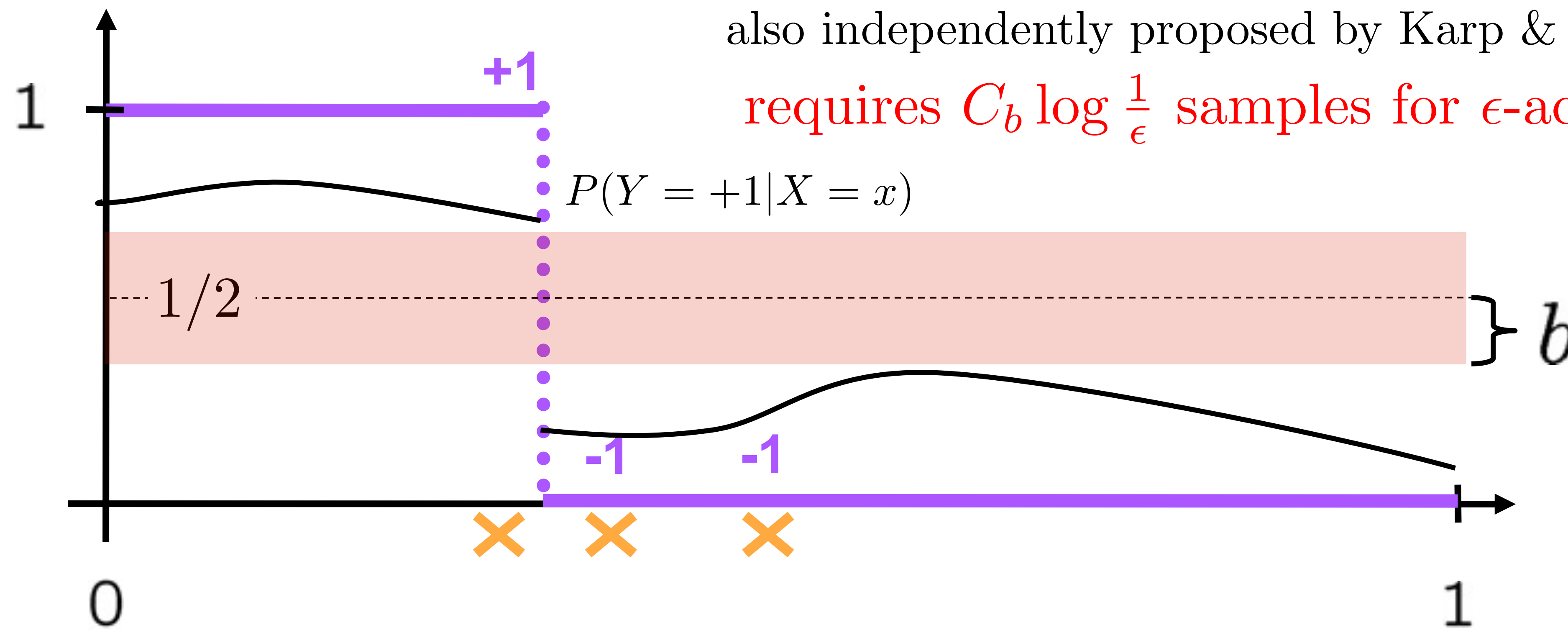


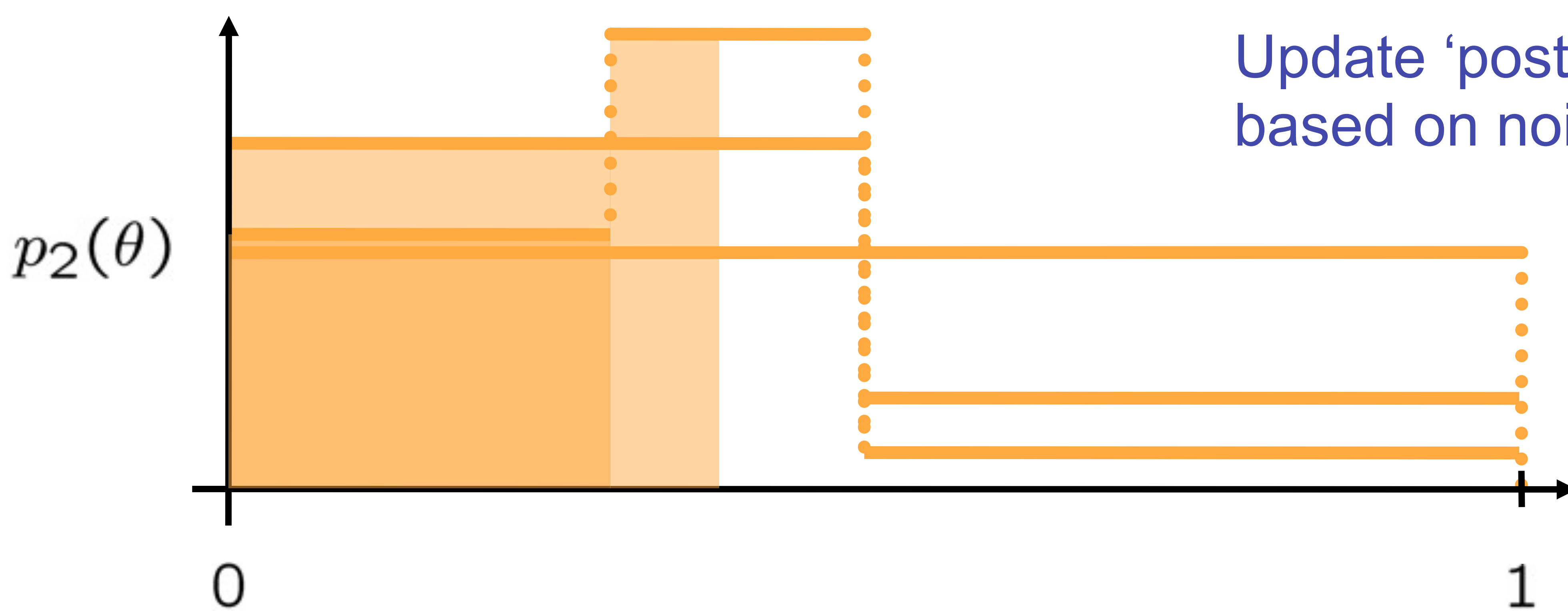
ϵ-accuracy requires $O\left(\frac{1}{\epsilon}\right)$ queries

Dealing with Noise (Horstein's Algorithm)

see **Burnashev & Zigangirov '74** for rigorous analysis;
also independently proposed by Karp & Kleinberg '07

requires $C_b \log \frac{1}{\epsilon}$ samples for ϵ-accuracy





Update 'posterior' density
based on noise bound b

Multidimensional Generalizations

Noisy Generalized Binary Search

initialize: p_0 uniform over $\mathcal{H}$ and $\alpha < \beta < 1/2$.

for $n = 0, 1, 2, \ldots$

1) $x_n = \arg\min_{x \in \mathcal{X}} \left|\sum_{h \in \mathcal{H}} p_n(h) h(x)\right|$

2) Obtain noisy response y_n

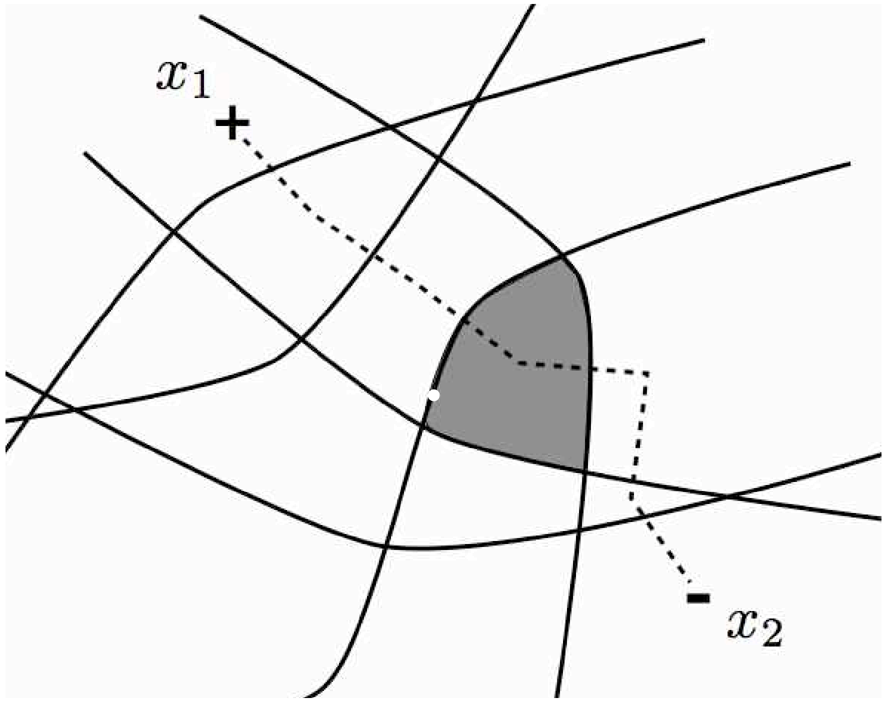
3) Bayes update: $\forall\, h$

$$p_{n+1}(h) \;\propto\; p_n(h) \times \begin{cases} 1-\beta & , h(x_n) = y_n \\ \beta & , h(x_n) \neq y_n \end{cases}$$

hypothesis selected at each step:

$\widehat{h}_n := \arg\max_{h \in H} p_n(h)$

"generalized" binary search is similar to classic binary search



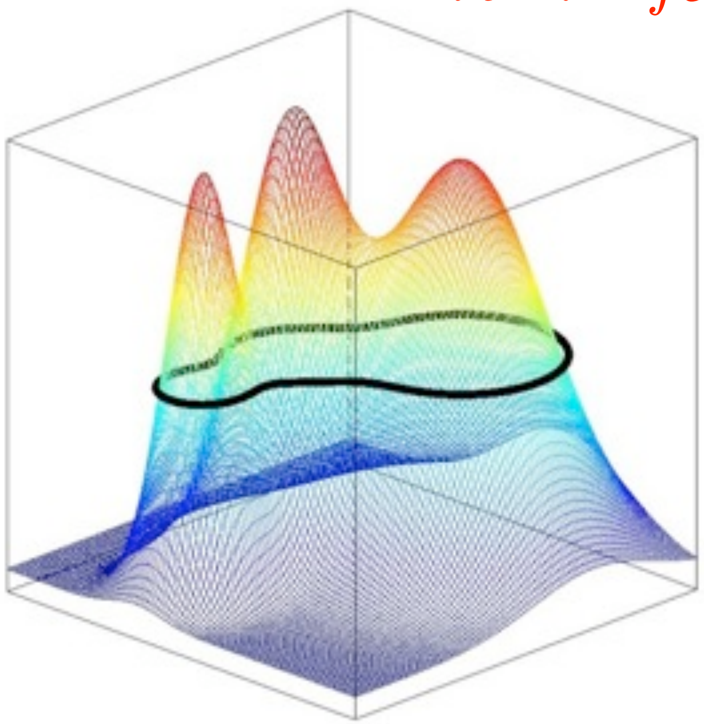
also requires as few as $\log \frac{1}{\epsilon}$ samples for ϵ-accuracy

... but more in general, depending on complexity of optimal decision boundary and noise characteristics

RN. Information Theory, IEEE Transactions on, Vol. 57, No. 12. (December 2011), pp. 7893-7906.

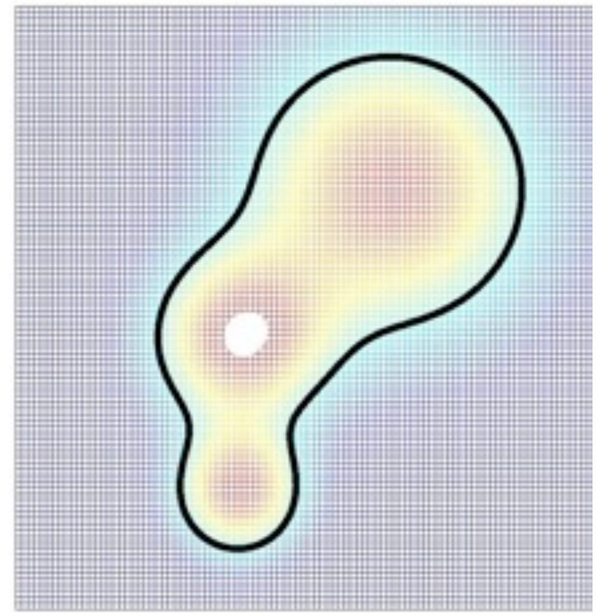
Nonparametric Binary Classification

$\mathcal{X}$:= *feature* space, typically $\mathbb{R}^d$ $\mathcal{Y} := \{-1, +1\}$, labels

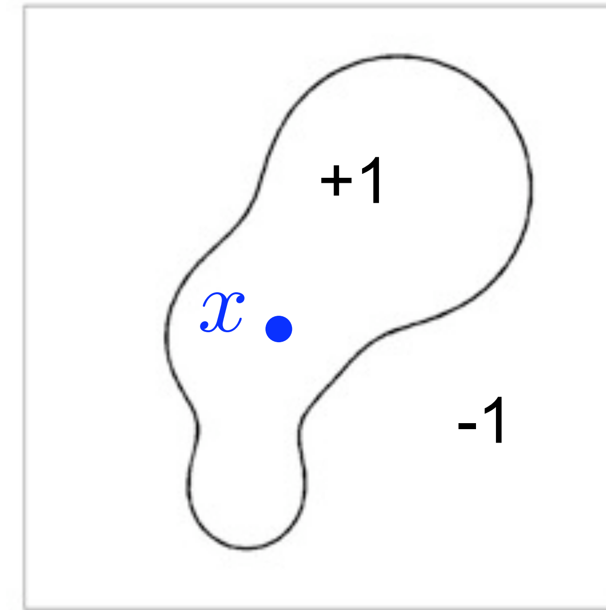


$\mathbb{P}(Y = 1|X = x)$

unknown



1/2-level set is optimal decision boundary



optimal decision set

allowable questions: is x in the set?

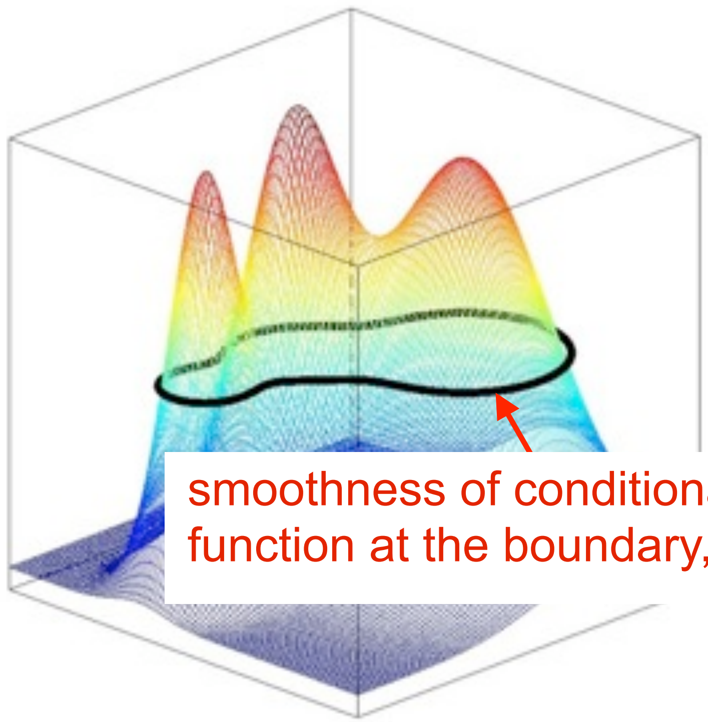
Key Questions:

1. When can active learning provide reductions in sample complexity?
2. What active learning strategies/policies are optimal?

Castro and N (2008), Raginsky and Rahklin (2011), Hanneke (2011)

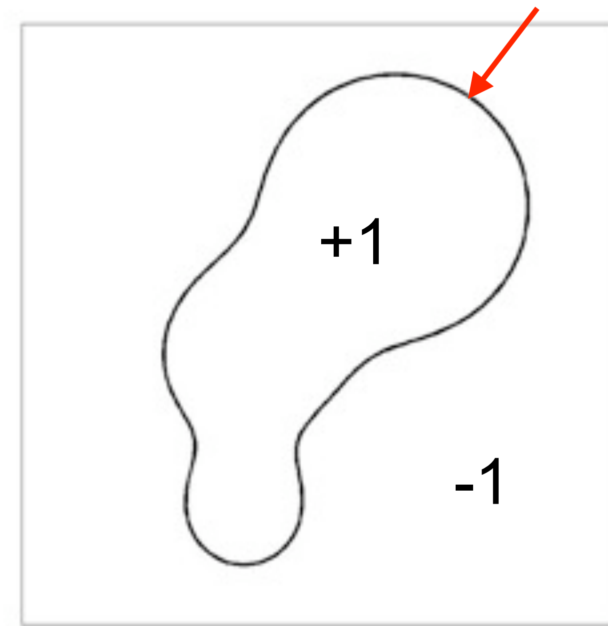
Bounds on Sample Complexity

Key complexity parameters



$\mathbb{P}(Y = 1 | X = x)$

Holder regularity of the decision boundary, α



optimal decision set

training examples: $\{(x_i, y_i)\}_{i=1}^n$ selected sequentially and adaptively (active learning) or at random (passive learning)

minimax rate of convergence to Bayes error:

Active: $\quad n^{-\frac{\kappa}{2\kappa+\rho-2}}$

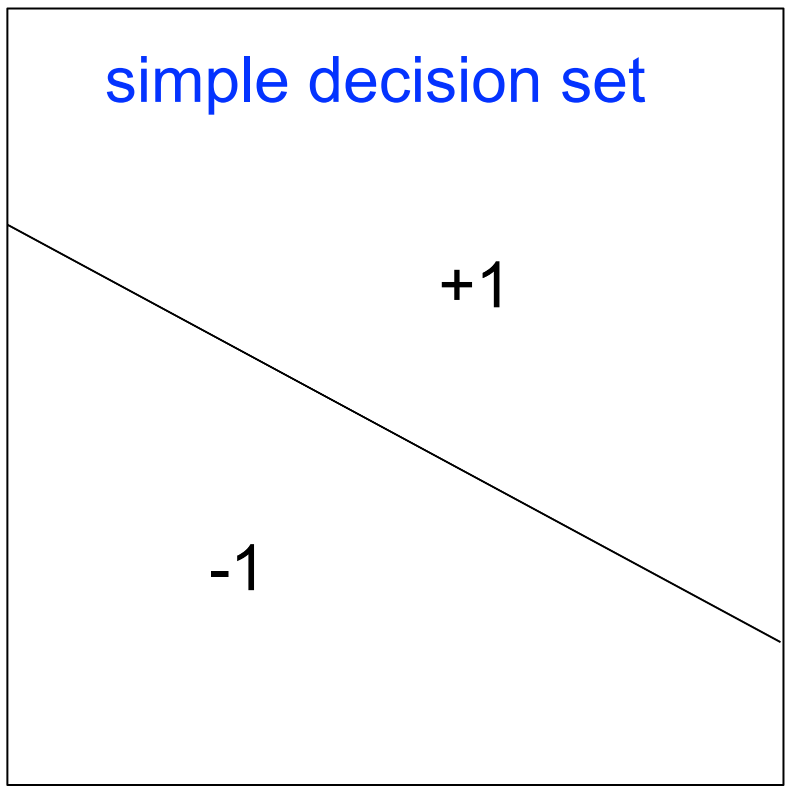
Passive: $\quad n^{-\frac{\kappa}{2\kappa+\rho-1}}$

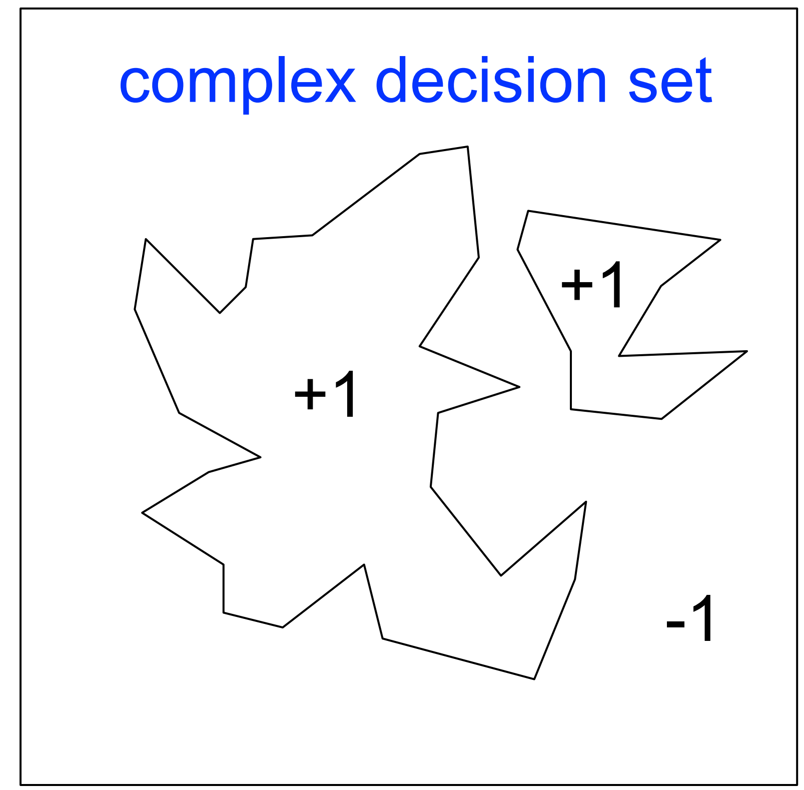
$$\rho := \frac{d-1}{\alpha}$$

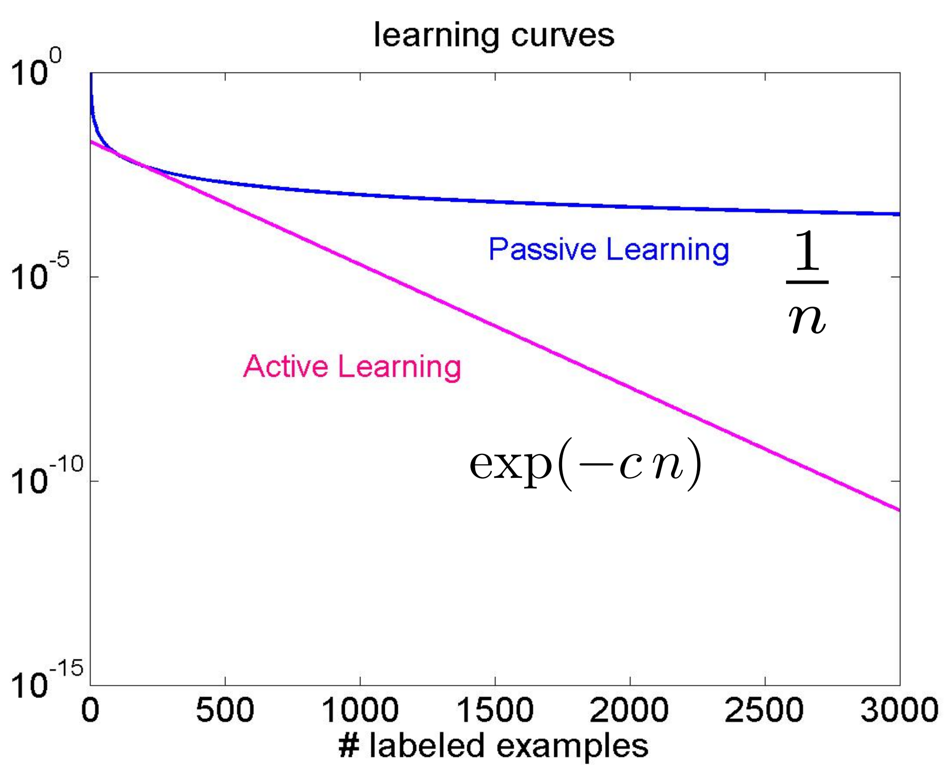
as $\rho \to 0$
and $\kappa \to 1$

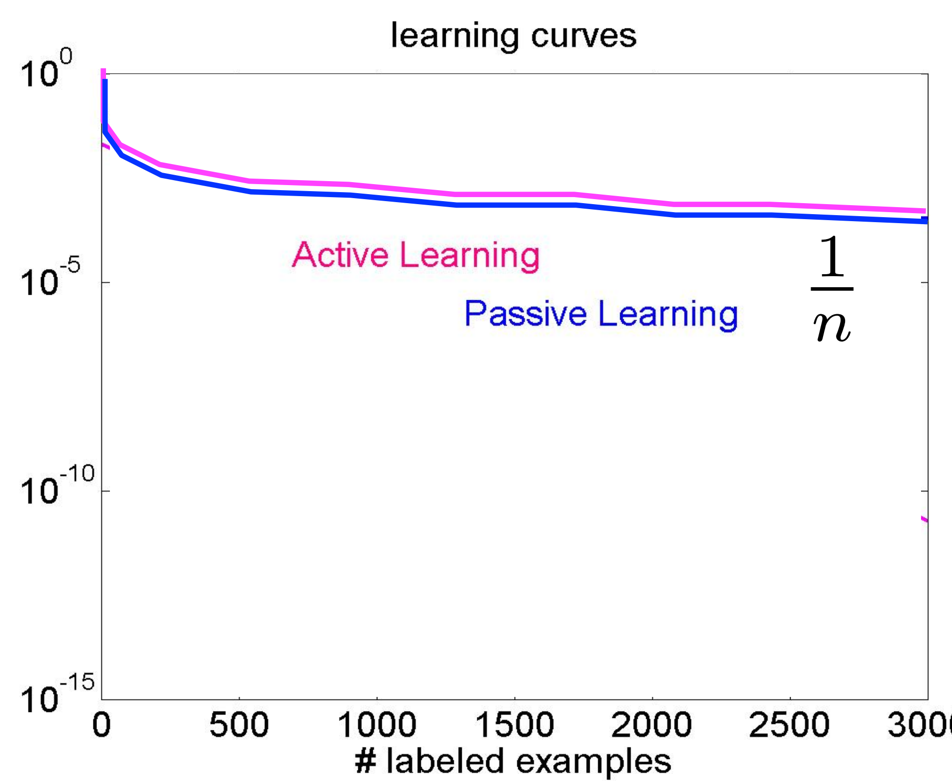
active learning yields exponential improvement!

Implications in Practice









Ranking

Bartender: "What beer would you like?"

Jeff: "Hmm... I usually drink Duff"

Bartender: "Try these two samples. Do you prefer A or B?

Jeff: "B"

Bartender: "Ok try these two: C or D?" ....

Ranking Based on Pairwise Comparisons

Consider 10 beers ranked from best to worst: D < G < I < C < J < E < A < H < B < F

	A	B	C	D	E	F	G	H	I	J
A	0	1	-1	-1	-1	1	-1	1	-1	-1
B	-1	0	-1	-1	-1	1	-1	-1	-1	-1
C	1	1	0	-1	1	1	-1	1	-1	1
D	1	1	1	0	1	1	1	1	1	1
E	1	1	-1	-1	0	1	-1	1	-1	-1
F	-1	-1	-1	-1	-1	0	-1	-1	-1	-1
G	1	1	1	-1	1	1	0	1	1	1
H	-1	1	-1	-1	-1	1	-1	0	-1	-1
I	1	1	1	-1	1	1	-1	1	0	1
J	1	1	-1	-1	1	1	-1	1	-1	0

Which questions should we ask? How many are needed?

Does adaptively help?

Randomly Selected Pairwise Comparisons

Consider 10 beers ranked from best to worst:

D < G < I < C < J < E < A < H < B < F

	A	B	C	D	E	F	G	H	I	J
A	0	1	-1	-1	-1	1	-1	1	-1	-1
B	-1	0	-1	-1	-1	1	-1	-1	-1	-1
C	1	1	0	-1	1	1	-1	1	-1	1
D	1	1	1	0	1	1	1	1	1	1
E	1	1	-1	-1	0	1	-1	1	-1	-1
F	-1	-1	-1	-1	-1	0	-1	-1	-1	-1
G	1	1	1	-1	1	1	0	1	1	1
H	-1	1	-1	-1	-1	1	-1	0	-1	-1
I	1	1	1	-1	1	1	-1	1	0	1
J	1	1	-1	-1	1	1	-1	1	-1	0

select m pairwise comparisons **at random**

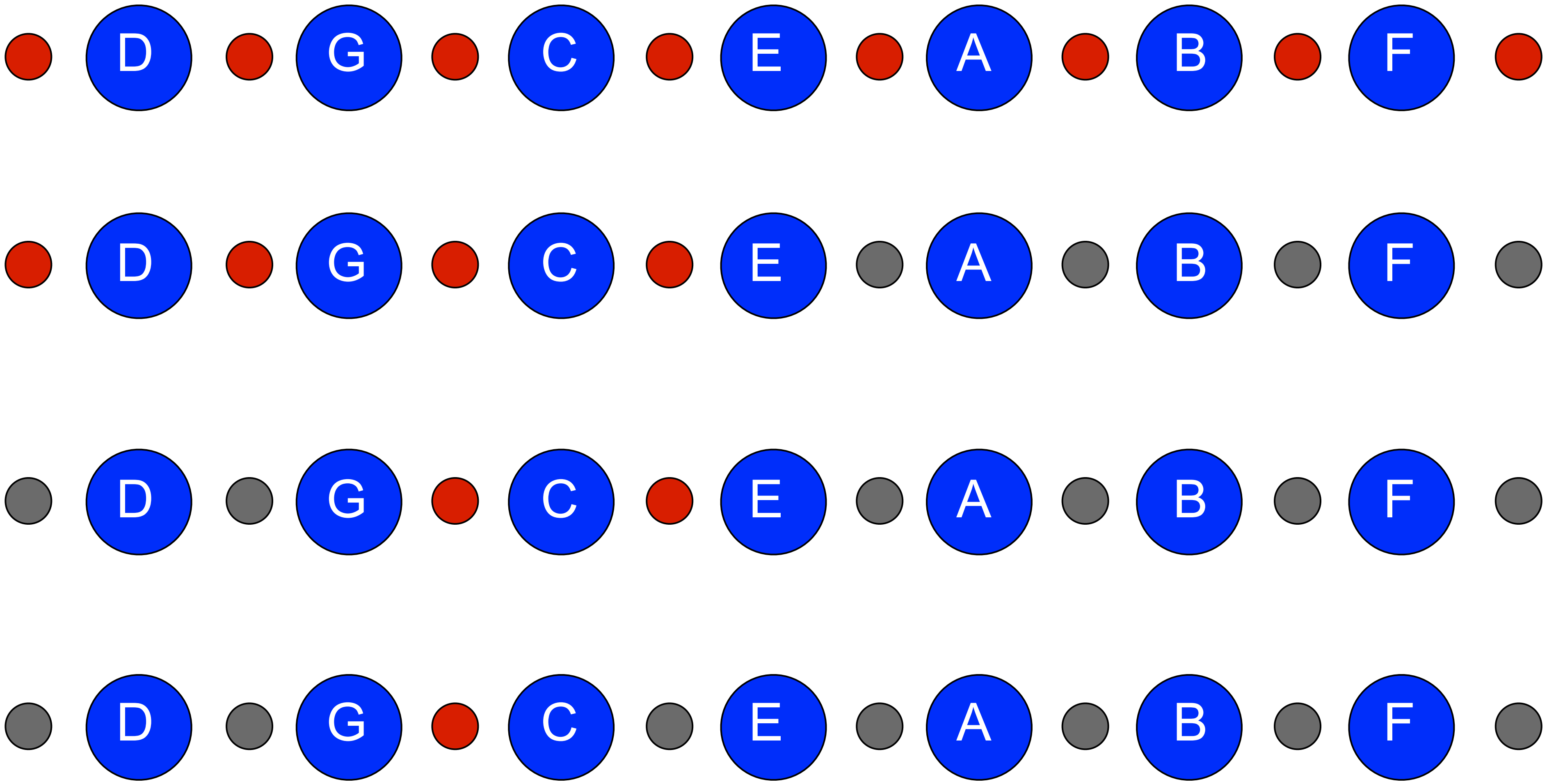
perfect recovery: almost all pairs must be compared, i.e., about $n(n-1)/2$ comparisons

approximate recovery: fraction of pairs misordered $\leq \dfrac{c\,n\log n}{m}$

That's a lot of beer!

Ranking with Adaptively Selected Queries

Insert H into: D < G < C < E < A < B < F



{ }

{H < E}

{H < E},{G < H}

{H < E},{G < H},{H < C}

D < G < H < C < E < A < B < F

to correctly place an object into an ordered list of k objects requires $\log_2 k$ comparisons

Adaptively Selected Pairwise Comparisons

Consider 10 beers ranked from best to worst:

D < G < I < C < J < E < A < H < B < F

	A	B	C	D	E	F	G	H	I	J
A	0	1	-1	-1	-1	1	-1	1	-1	-1
B	-1	0	-1	-1	-1	1	-1	-1	-1	-1
C	1	1	0	-1	1	1	-1	1	-1	1
D	1	1	1	0	1	1	1	1	1	1
E	1	1	-1	-1	0	1	-1	1	-1	-1
F	-1	-1	-1	-1	-1	0	-1	-1	-1	-1
G	1	1	1	-1	1	1	0	1	1	1
H	-1	1	-1	-1	-1	1	-1	0	-1	-1
I	1	1	1	-1	1	1	-1	1	0	1
J	1	1	-1	-1	1	1	-1	1	-1	0

select m pairwise comparisons according to **binary sort**

Binary insertion sort: perfect recovery if

$$\log_2 k \text{ comparisons to insert an item into a list of k objects}$$
$$\implies n \log_2 n \text{ comparisons to rank } n \text{ objects}$$

That's still a lot of beer!

Beer Space

Suppose beers can be embedded (according to characteristics) into a low-dimensional Euclidean space.

A

B

W

Jeff's latent preferences in "beer space"
(e.g, bitterness, color, maltiness,...)

C

$$\|x_i - W\| < \|x_j - W\| \;\Leftrightarrow\; x_i \prec x_j$$

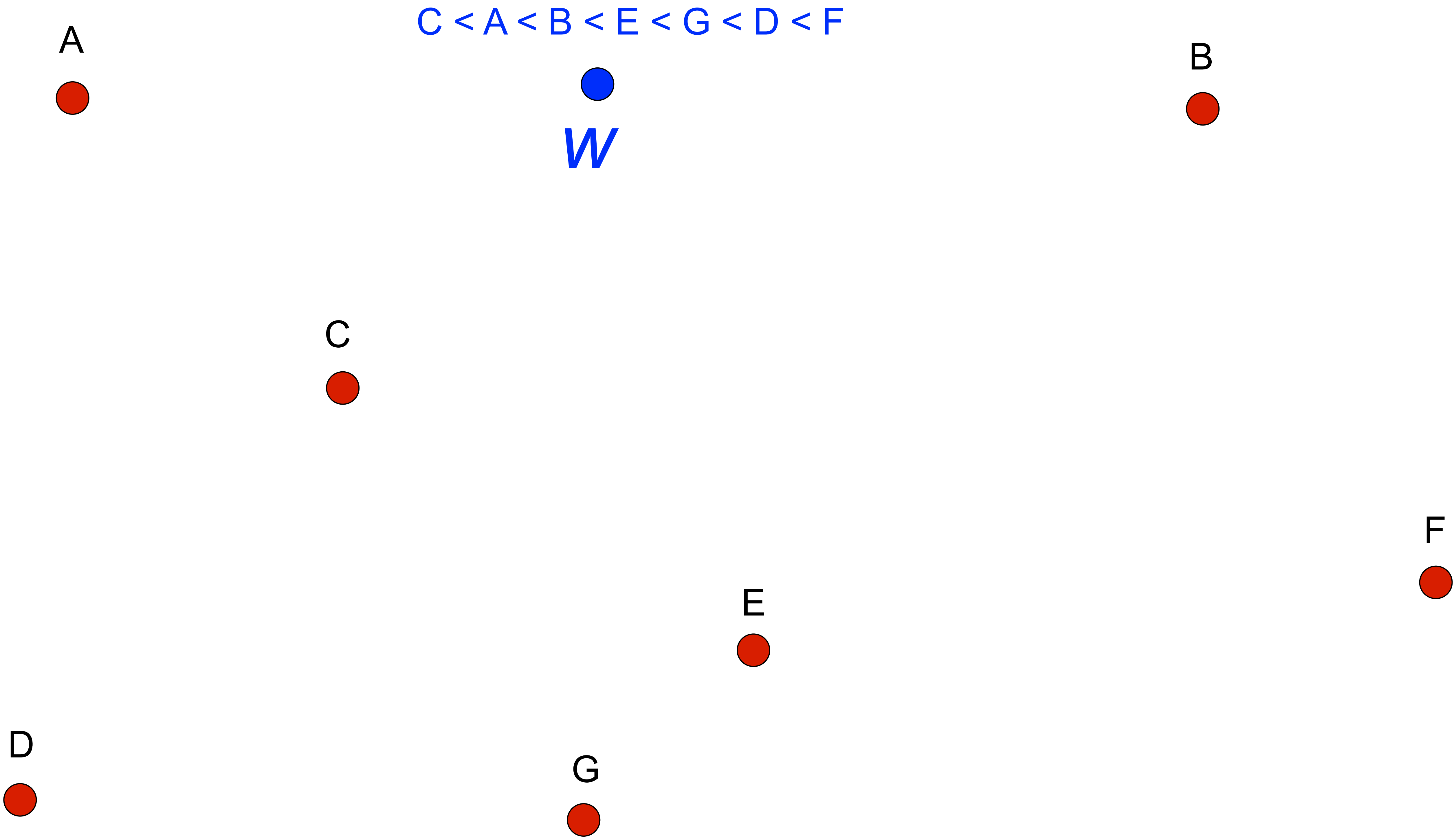
F

E

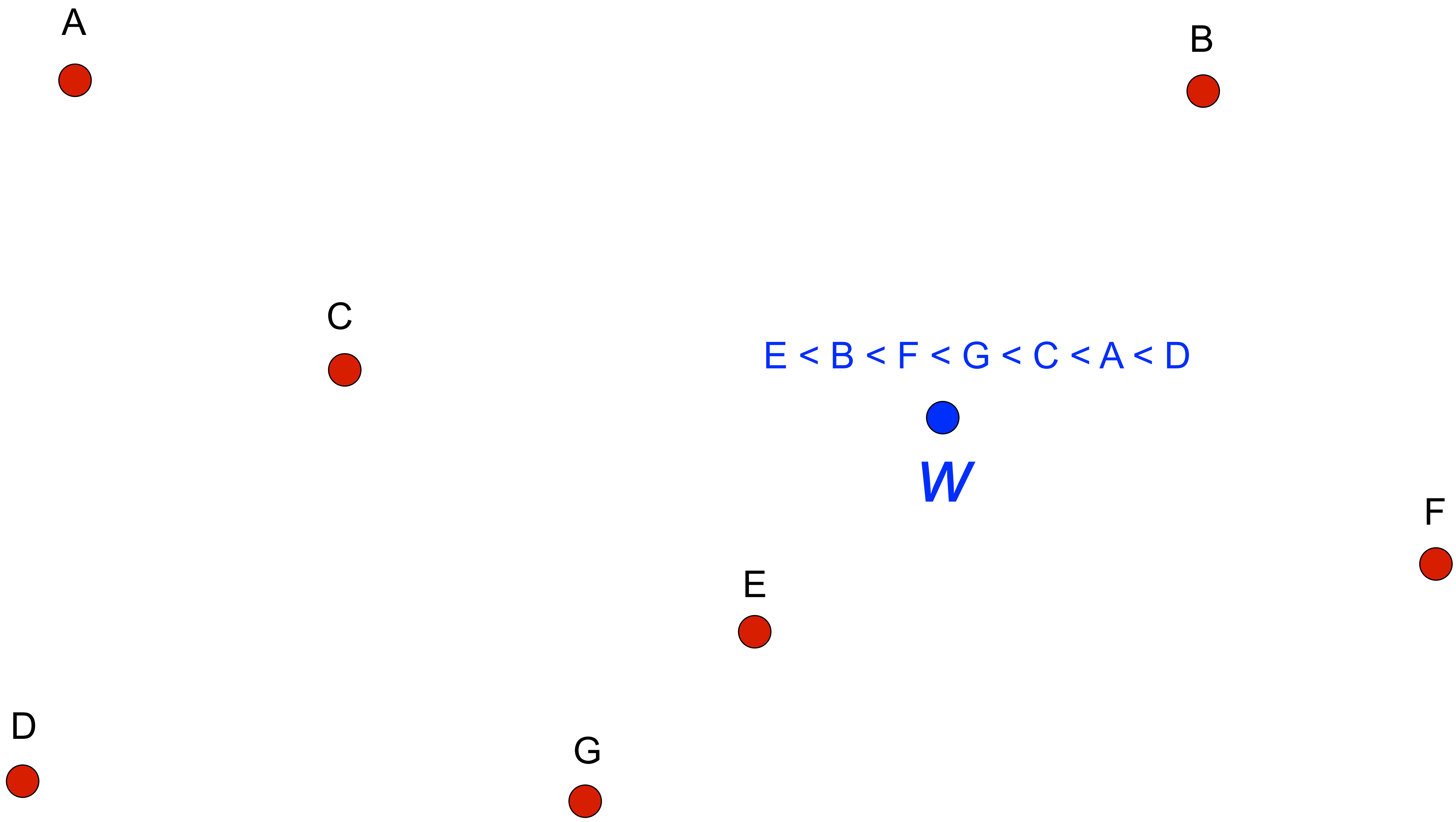
D

G

Ranking According to Distance



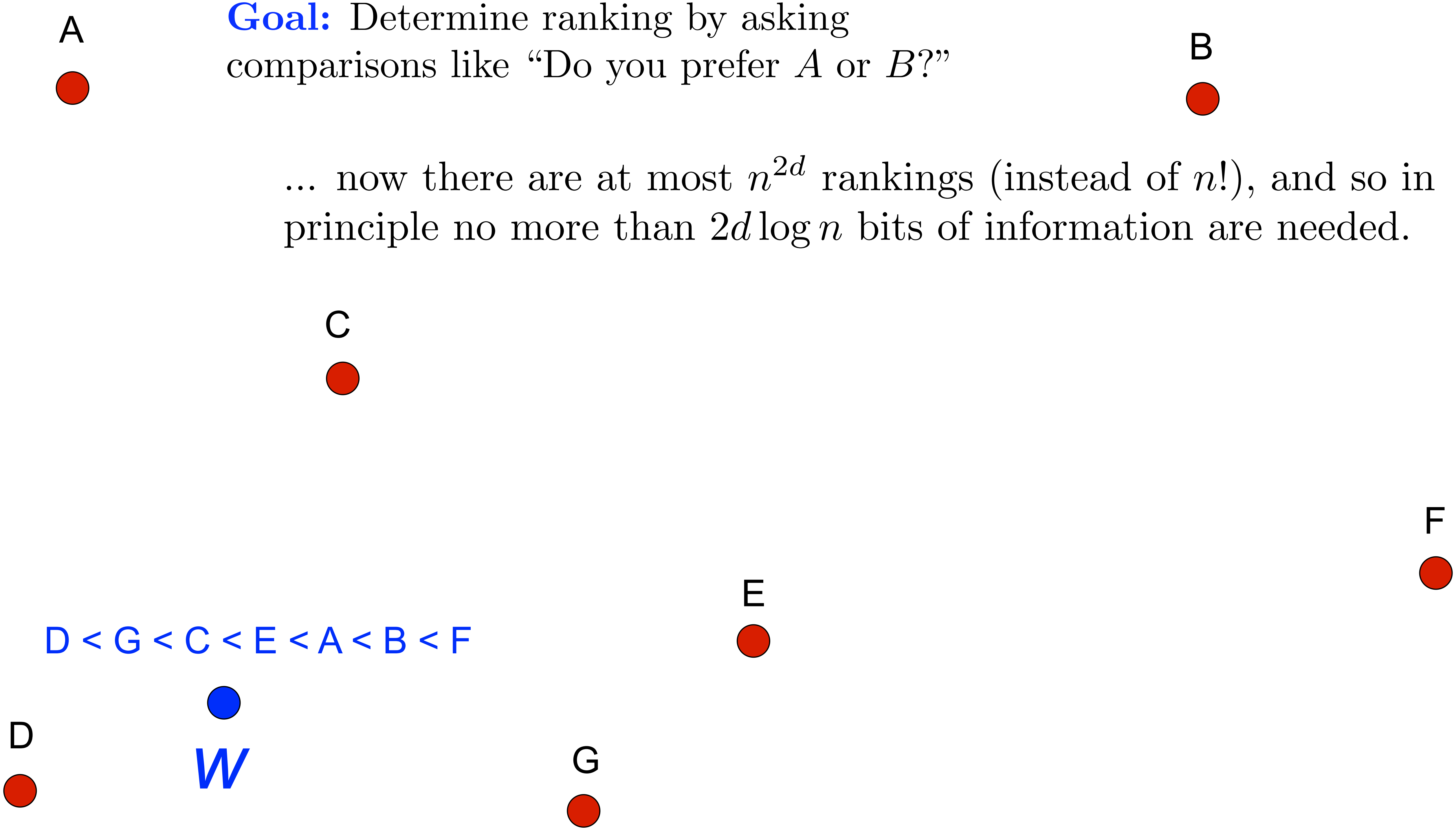
Ranking According to Distance



Ranking According to Distance

Goal: Determine ranking by asking comparisons like "Do you prefer A or B?"

... now there are at most n^{2d} rankings (instead of $n!$), and so in principle no more than $2d \log n$ bits of information are needed.



Lazy Binary Search

Consider n objects $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$. Many comparisons are redundant because the objects embed in $\mathbb{R}^d$, and therefore it may be possible to correctly rank based on a small subset.

binary information we can gather: $q_{i,j} \equiv$ **do you prefer x_i or x_j**

Optimal selection of a sequence of $q_{i,j}$ requires a computationally difficult search, involving a combinatorial optimization.

Lazy Binary Search

input: $x_1, \ldots, x_n \in \mathbb{R}^d$
initialize: $x_1, \ldots, x_n$ in uniformly random order

for k=2,...,n
  for i=1,...,k-1
    if $q_{i,k}$ is ***ambiguous*** given $\{q_{i,j}\}_{i,j<k}$,
      then ask for pairwise comparison,
    else impute $q_{i,j}$ from $\{q_{i,j}\}_{i,j<k}$

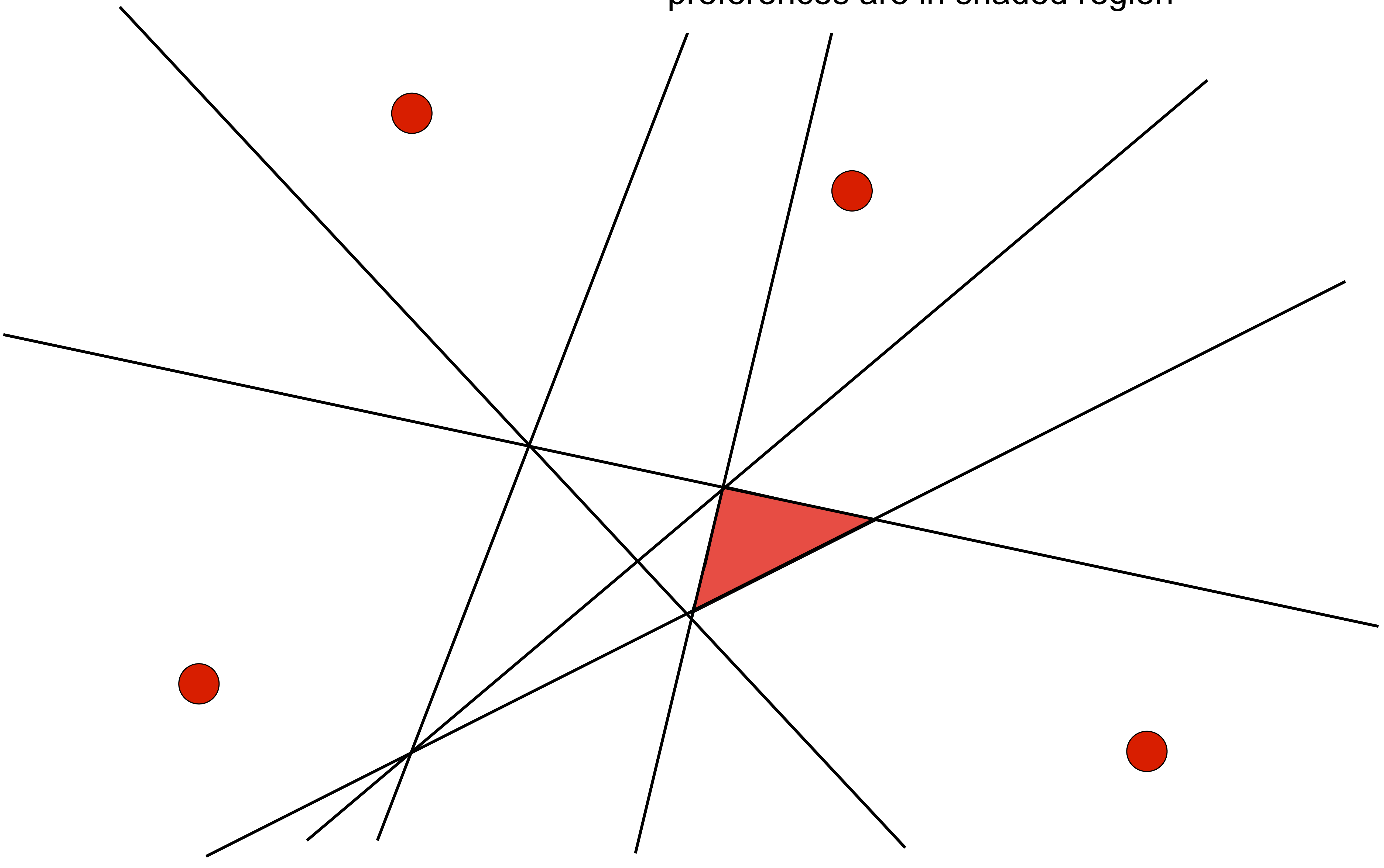
output: ranking of $x_1, \ldots, x_n$ consistent with *all* pairwise comparisons

simple linear program

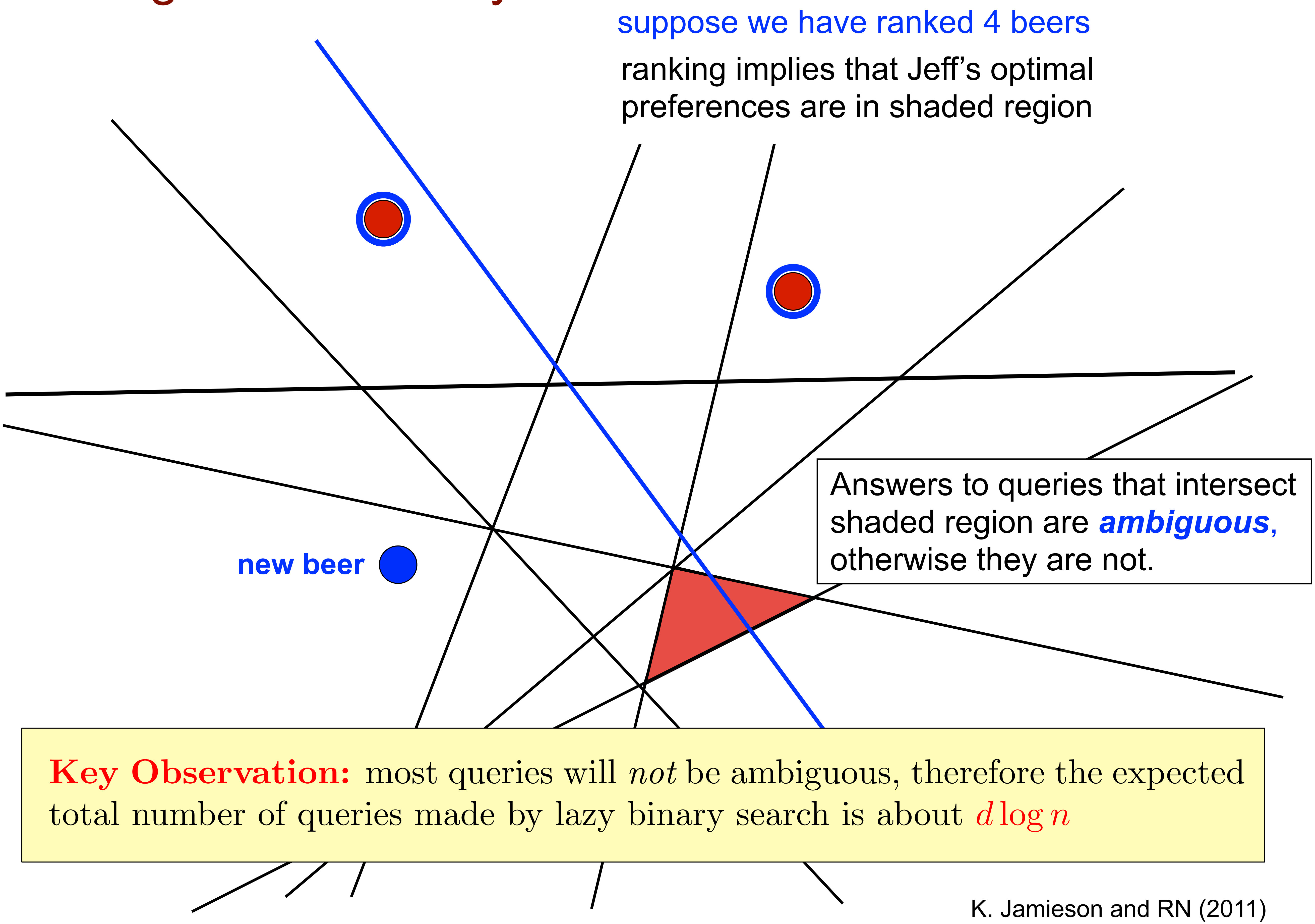
Ranking and Geometry

suppose we have ranked 4 beers

ranking implies that Jeff's optimal preferences are in shaded region



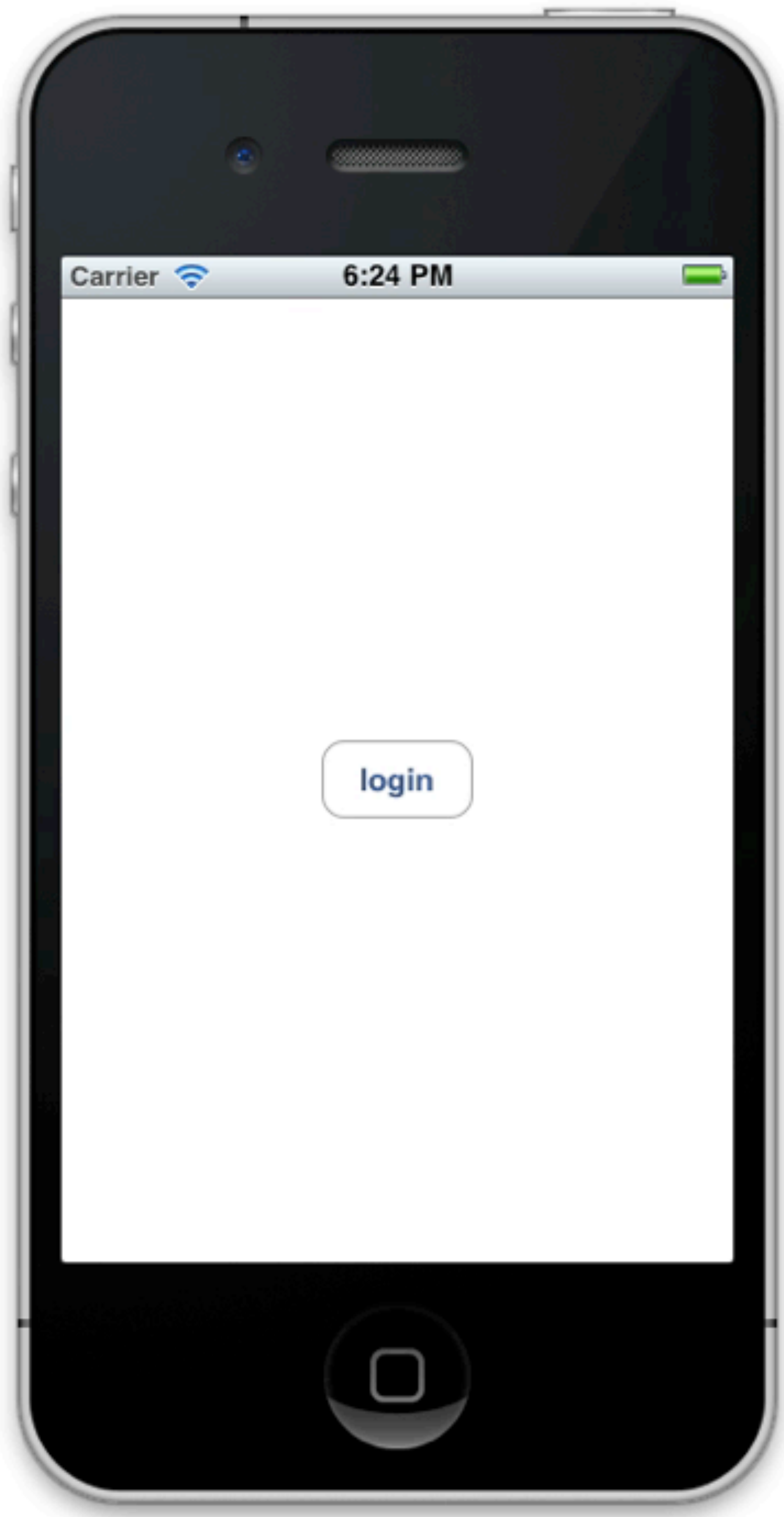
Ranking and Geometry



Key Observation: most queries will *not* be ambiguous, therefore the expected total number of queries made by lazy binary search is about $d \log n$

K. Jamieson and RN (2011)

BeerMapper



BeerMapper app learns a persons ranking of beers by selecting pairwise comparisons using lazy binary search and a low-dimensional embedding based on key beer features

BeerMapper - Under the Hood

Algorithm requires feature representations of the beers $\{x_1, \ldots, x_n\} \subset \mathbb{R}^d$

ratebeer

Two Hearted Ale - Input ~2500 natural language reviews

http://www.ratebeer.com/beer/two-hearted-ale/1502/2/1/

3.8 AROMA 8/10 APPEARANCE 4/5 TASTE 8/10 PALATE 3/5 OVERALL 15/20
fonefan (25678) - VestJylland, DENMARK - JAN 18, 2009

Bottle 355ml.
Clear light to medium yellow orange color with a average, frothy, good lacing, fully lasting, off-white head. Aroma is moderate to heavy malty, moderate to heavy hoppy, perfume, grapefruit, orange shell, soap. Flavor is moderate to heavy sweet and bitter with a average to long duration. Body is medium, texture is oily, carbonation is soft. [250908]

4 AROMA 8/10 APPEARANCE 4/5 TASTE 7/10 PALATE 4/5 OVERALL 17/20
Ungstrup (24358) - Oamaru, NEW ZEALAND - MAR 31, 2005

An orange beer with a huge off-white head. The aroma is sweet and very freshly hoppy with notes of hop oils - very powerful aroma. The flavor is sweet and quite hoppy, that gives flavors of oranges, flowers as well as hints of grapefruit. Very refreshing yet with a powerful body.

Reviews for each beer	Bag of Words weighted by TF*IDF	Get 15 nearest neighbors using cosine distance	Non-metric multidimensional scaling	Embedding in 3 dimensions

BeerMapper - Under the Hood

Algorithm requires feature representations of the beers $\{x_1, \ldots, x_n\} \subset \mathbb{R}^d$

Two Hearted Ale - Weighted Bag of Words (sorted by weights):

ipa hops citrus floral orange pine grapefruit head hoppy aroma white pours bitter golden piney hazy balanced cloudy malt amber sweet lacing bells strong light favorite gold off medium perfect hearted nose thick smooth excellent huge smell wonderful crisp poured fresh beautiful lots bell's creamy body copper flavors smells slightly fruity love yellow ever there amazing notes fluffy clean frothy sweetness brew long awesome ale caramel aromas flowers lemon palate malts over down get after tastes mouthfeel your backbone dry other leaves centennial top slight bite solid again batch right nicely through clear it's extremely foamy aftertaste still

Reviews for each beer	Bag of Words weighted by TF*IDF	Get 15 nearest neighbors using cosine distance	Non-metric multidimensional scaling	Embedding in 3 dimensions

BeerMapper - Under the Hood

Algorithm requires feature representations of the beers $\{x_1, \ldots, x_n\} \subset \mathbb{R}^d$

Weighted count vector for the ith beer:

$$z_i \in \mathbb{R}^{400{,}000}$$

Cosine distance:

$$d(z_i, z_j) = 1 - \frac{z_i^T z_j}{||z_i||\,||z_j||}$$

<u>Two Hearted Ale - Nearest Neighbors:</u>

Bear Republic Racer 5
Avery IPA
Stone India Pale Ale (IPA)
Founders Centennial IPA
Smuttynose IPA
Anderson Valley Hop Ottin IPA
AleSmith IPA
BridgePort IPA
Boulder Beer Mojo IPA
Goose Island India Pale Ale
Great Divide Titan IPA
New Holland Mad Hatter Ale
Lagunitas India Pale Ale
Heavy Seas Loose Cannon Hop3
Sweetwater IPA

Reviews for each beer

Bag of Words weighted by TF*IDF

Get 15 nearest neighbors using cosine distance

Non-metric multidimensional scaling

Embedding in 3 dimensions

BeerMapper - Under the Hood

Algorithm requires feature representations of the beers $\{x_1, \ldots, x_n\} \subset \mathbb{R}^d$

Weighted count vector for the ith beer:

$$z_i \in \mathbb{R}^{400{,}000}$$

Cosine distance:

$$d(z_i, z_j) = 1 - \frac{z_i^T z_j}{||z_i||\,||z_j||}$$

Two Hearted Ale - Nearest Neighbors:

Bear Republic Racer 5
Avery IPA
Stone India Pale Ale (IPA)
Founders Centennial IPA
Smuttynose IPA
Anderson Valley Hop Ottin IPA
AleSmith IPA
BridgePort IPA
Boulder Beer Mojo IPA
Goose Island India Pale Ale
Great Divide Titan IPA
New Holland Mad Hatter Ale
Lagunitas India Pale Ale
Heavy Seas Loose Cannon Hop3
Sweetwater IPA

Reviews for each beer

Bag of Words weighted by TF*IDF

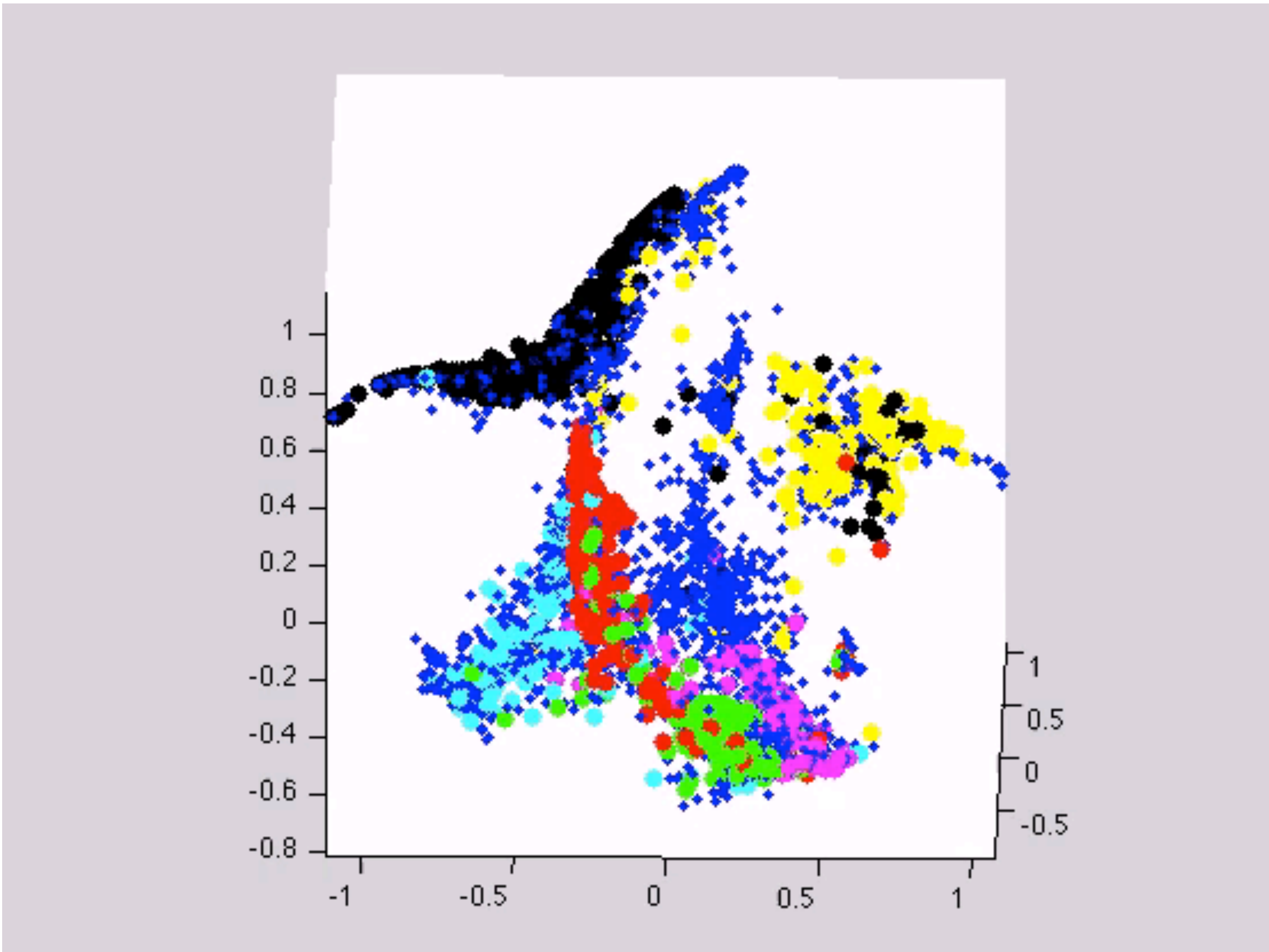
Get 15 nearest neighbors using cosine distance

Non-metric multidimensional scaling

Embedding in 3 dimensions

BeerMapper - Under the Hood

Algorithm requires feature representations of the beers $\{x_1, \ldots, x_n\} \subset \mathbb{R}^d$



Sanity check: styles should cluster together and similar styles should be close.

Red = IPA
Green = Pale Ale
Magenta = Amber Ale
Cyan = Lager + Pilsener
Yellow = Belgians (light + dark)
Black = Stout + Porter
Blue = Everything else

Reviews for each beer

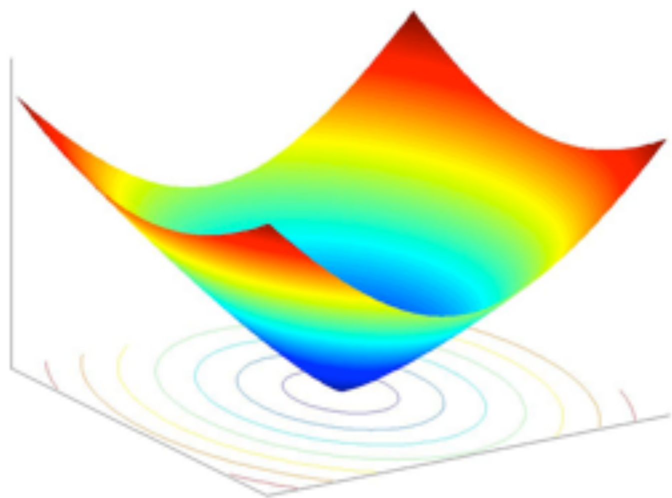
Bag of Words weighted by TF*IDF

Get 15 nearest neighbors using cosine distance

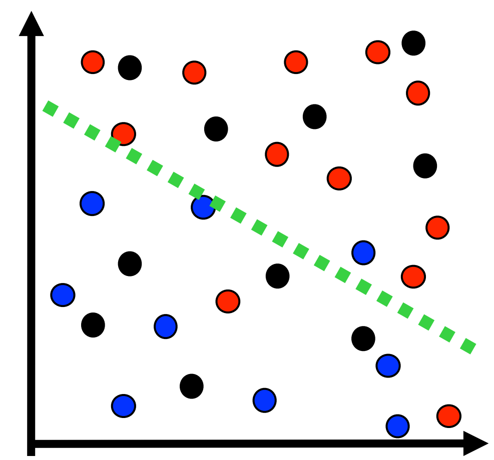
Non-metric multidimensional scaling

Embedding in 3 dimensions

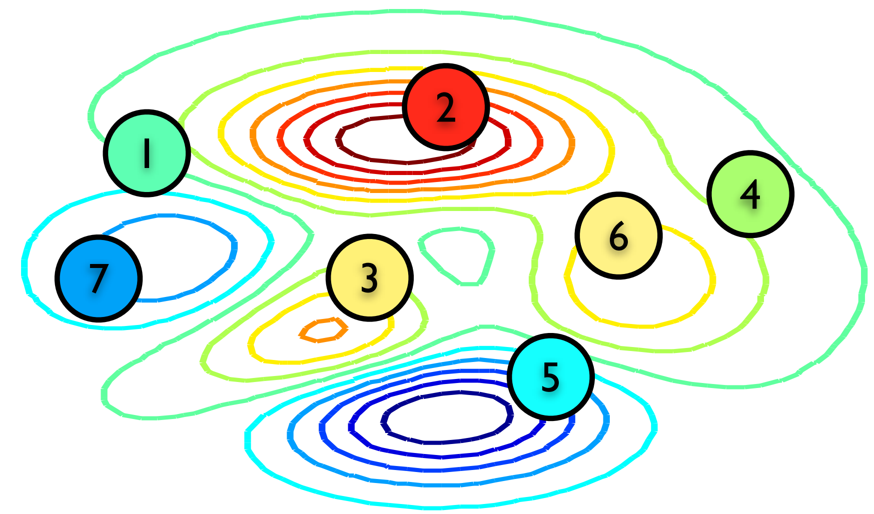
Machine Learning from Human Judgements



Derivative Free Optimization
using Human Subjects



Binary Classification
via Active Learning



Ranking from
Pairwise Comparisons

Challenge:

Computing is cheap, but human assistance/guidance is expensive

Goal:

Optimize such systems with as little human involvement as possible

Humans are much more reliable and consistent at making comparative judgements, than in giving numerical ratings or evaluations

“Binary search” procedures can play a role in *active learning*

References

J. Haupt, R. Castro, and R. Nowak, “Distilled sensing,” IEEE Trans. IT 2011

J. Haupt, R. Castro, R. Baraniuk, and R. Nowak, “Sequentially designed compressed sensing,” SSP 2012

T. Bijmolt and M. Wedel, “The effects of alternative methods of collecting similarity data for multidimensional scaling,” IJRM 1995

N. Steward, G. Brown and N. Chater, “Absolute identification by relative judgement,” Psych. Review 2005

K. Jamieson, B. Recht, and R. Nowak, “Query complexity of derivative free optimization,” arxiv 2012

A. Agrawal, O. Dekel and L. Xiao, “Optimal algorithms for online convex optimization with multi-point bandit feedback,” COLT 2010

A. Nemirovski, A. Juditsky, G. Lan and A. Shapiro, “Robust stochastic approximation approach to stochastic programming,” SIAM J. Opt 2009

S. Tong and D. Koller, “Support vector machine active learning with applications,” JMLR 2001

M. Horstein, “Sequential decoding using noiseless feedback,” IEEE Trans. IT 1963

M. Burnashev and K. Zigangirov, “An interval estimation problem for controlled observations,” Prob. Info. Transmission 1974

R. Karp and R. Kleinberg, “Noisy binary search and its applications,” SODA 2007

R. Nowak, “The geometry of generalized binary search,” IEEE Trans. IT 2011

R. Castro and R. Nowak, “Minimax bounds for active learning,” IEEE Trans. IT 2008

S. Hanneke, “Rates of convergence in active learning,” Ann. Stat. 2011

M. Raginsky and S. Rahklin, “Lower bounds for passive and active learning,” NIPS 2011

K. Jamieson and R. Nowak, “Active ranking using pairwise comparisons,” NIPS 2011